1. Solve each of the following differential equations:

(a) \( y' + 3xy = 0 \)
(b) \( y' + 3y = 3x \)
(c) \( \frac{dy}{dt} = \cos(t) y \)
(d) \( x^2 \frac{dy}{dx} - y = 3 \)

Solution:

(a) This is a first-order linear equation, so we use an integrating factor. The integrating factor is \( \rho(x) = e^{3x^2/2} \), so we have

\[
e^{3x^2/2}y'(x) + 3xe^{3x^2/2}y(x) = 0,
\]

\[
\frac{d}{dx}(e^{3x^2/2}y(x)) = 0,
\]

\[
e^{3x^2/2}y(x) = C,
\]

\[
y(x) = Ce^{-3x^2/2}.
\]

(We could also use separation.)

(b) Again first-order linear, and our integrating factor is \( \rho(x) = e^{3x} \). So we have

\[
e^{3x}y' + 3e^{3x}y = 3xe^{3x},
\]

\[
\frac{d}{dx}(e^{3x}y) = 3xe^{3x},
\]

\[
e^{3x}y = xe^{3x} - \frac{1}{3}e^{3x} + C,
\]

\[
y(x) = x - \frac{1}{3} + Ce^{-3x}.
\]

(c) We separate variables to get

\[
\frac{dy}{y} = \cos(t) \, dt,
\]

or

\[
\log |y| = \sin(t) + C,
\]

or

\[
y(t) = Ce^{\sin t}.
\]

(d) We first divide by \( x^2 \) and get

\[
y' - \frac{1}{x^2}y = \frac{3}{x^2}.
\]

Our integrating factor should be \( e^{1/x} \) and so we get

\[
e^{1/x}y' - \frac{1}{x^2}e^{1/x}y = \frac{3}{x^2}e^{1/x},
\]

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Integrating gives
\[ e^{1/x} y(x) = -3e^{1/x} + C, \]
or
\[ y(x) = -3 + Ce^{1/x}. \]

2. Solve each of the following IVPs:
   (a) \( y' - 2y = 0, \quad y(0) = 0 \)
   (b) \( \frac{dy}{dx} = x^2 y, \quad y(1) = 2 \)
   (c) \( (1 + x)y' + y = 3, \quad y(0) = -1 \)
   (d) \( xy' + (3x + 1)y = 5, \quad y(2) = 4 \)

**Solution:**

(a) This one we can solve many ways, but we see that the solution is \( y(x) = Ce^{2x} \). Plugging in the initial condition gives \( C = 0 \), so the solution is \( y(x) \equiv 0 \).

(b) We solve by separation to get
\[
\frac{dy}{y} = x^2 \, dx,
\]
\[
\ln |y| = \frac{x^3}{3} + C,
\]
\[
y(x) = Ce^{x^3/3}.
\]

Plugging in the initial condition gives \( y(1) = Ce^{1/3} = 2 \), so \( C = 2e^{-1/3} \), and thus the solution is
\[
y(x) = 2e^{(x^3 - 1)/3}.
\]

(c) We see that the left-hand side is already in the form of a product rule, so we can write
\[
\frac{d}{dx}((1 + x)y(x)) = 3
\]
\[
(1 + x)y(x) = 3x + C
\]
\[
y(x) = \frac{3x}{1 + x} + \frac{C}{1 + x}.
\]

(If we didn’t make this observation, then, first divide by the front coefficient to get
\[
y' + \frac{1}{1+x} y = \frac{3}{1 + x}.
\]
The integrating factor will be
\[
\rho(x) = e^{\int \frac{1}{1+x} \, dx} = e^{\ln|1+x|} = 1 + x
\]
and multiplying through gives the original equation.) In any case, plugging in \( x = 0 \) gives
\[
y(x) = C = -1,
\]
so the solution is
\[
y(x) = \frac{3x - 1}{1 + x}.
\]
(d) Divide through by the front coefficient to get
\[ y' + \left(3 + \frac{1}{x}\right)y = \frac{5}{x}. \]

The integrating factor is
\[ e^{\int \left(3 + \frac{1}{x}\right) \, dx} = e^{3x + \ln x} = xe^{3x}. \]

Multiplying through gives
\[ xe^{3x}y' + (3xe^{3x} + e^{3x})y = 5e^{3x}, \]
\[ \frac{d}{dx}(xe^{3x}y) = 5e^{3x}, \]
\[ xe^{3x}y(x) = \frac{5}{3}e^{3x} + C, \]
\[ y(x) = \frac{5}{3x} + \frac{C}{xe^{3x}}. \]

Plugging in \( x = 2 \) gives
\[ y(2) = \frac{5}{6} + \frac{C}{2e^6} = 4, \]
or
\[ C = \frac{19}{3}e^6. \]

This gives
\[ y(x) = \frac{5}{3x} + \frac{19}{3xe^{3x-6}}. \]

3. (Problem 1.5.36 from book.) A tank initially contains 60 gal of pure water. Salt water containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the perfectly mixed solution leaves the tank at 3 gal/min. Thus the tank is empty after 1 hour. Find the amount of salt in the tank after \( t \) minutes. Determine the maximum amount of salt ever in the tank.

**Solution:** First note that the tank is losing a net of 1 gal/min of liquid, which means that it will be empty in 60 minutes. More specifically, the number of gallons in the tank at time \( t \) is \( V(t) = 60 - t \).

Second, we want to write down a differential equation for the salt at time \( t \), which we will denote by \( S(t) \). Clearly, the rate of change of salt will be the rate coming in minus the rate going out. The amount coming in is 1 lb/gal * 2 gal/min = 2 lb/min. The rate going out is more complicated, since it depends on the concentration of the salt in the tank at the time. But notice that if there is \( S(t) \) pounds of salt at time \( t \), and there are 60 - \( t \) gallons in the tank at time \( t \), then the concentration is thus \( S(t)/(60 - t) \) lb/gallon in the tank. The rate of liquid leaving is 3 gal/min, so the rate out is \( 3S(t)/(60 - t) \). Thus we have
\[ \frac{dS}{dt} = 2 - \frac{3S(t)}{60 - t}, \]

or
\[ S' + \frac{3}{60 - t}S = 2. \]

We will use the integrating factor
\[ e^{\int \frac{3}{60 - t} \, dt} = e^{-3\ln(60-t)} = e^{\ln((60-t)^{-3})} = (60 - t)^{-3}. \]
Then we have

\[(60 - t)^{-3}S' + 3(60 - t)^{-4}S = 2(60 - t)^{-3},\]

\[
\frac{d}{dt}((60 - t)^{-3}S(t)) = 2(60 - t)^{-3},
\]

\[(60 - t)^{-3}S(t) = (60 - t)^{-2} + C,
\]

\[S(t) = C(60 - t)^3 + (60 - t).
\]

Now, we also need an initial condition, but notice that there is no salt in the tank at the start, so that \(S(0) = 0\). Plugging in gives

\[S(0) = C \cdot 60^3 + 60 = 0,
\]

or

\[C = -\frac{1}{60^2} = -\frac{1}{3600}.
\]

Thus the solution for all \(t\) between 0 and 60 minutes is given by

\[S(t) = -\frac{(60 - t)^3}{3600} + (60 - t).
\]

(We can check that \(S(60) = 0\), as it should.) Now we want to know the maximum value of \(S(t)\) for \(t\) between 0 and 60. We first compute its derivative

\[S'(t) = \frac{-(60 - t)^2}{1200} + 1.
\]

Setting this equal to zero gives

\[(60 - t)^2 = 1200
\]

or

\[t = 60 \pm \sqrt{1200}.
\]

Since we only care about \(t\) less than 60, we want to take the solution with the minus and not the plus. Thus the value we are looking for is \(S(60 - \sqrt{1200}) = 20(\sqrt{3} - 1/\sqrt{3}) \approx 23.094\).

4. Find constants \(A, B\) so that

\[y(x) = A \sin x + B \cos x
\]

is a solution of

\[y' + y = 4 \sin x.
\]

Now, find constants \(A, B, C\) so that

\[y(x) = A \sin x + B \cos x + Ce^{-x}
\]

is a solution to

\[y' + y = 4 \sin x, \quad y(0) = 4.
\]

**Solution:** We plug in. We start off with

\[y(x) = A \sin x + B \cos x,
\]

\[y'(x) = A \cos x - B \sin x
\]

So

\[y' + y = (A + B) \cos x + (A - B) \sin x,
\]
so we have

\[ A + B = 0, \]
\[ A - B = 1. \]

The solution to this is \( A = 1/2, B = -1/2 \), and our solution is

\[ y(x) = \frac{1}{2} (\sin x - \cos x). \]

As for the second, we first check to find \( A, B \):

\[ y(x) = A \sin x + B \cos x + C e^{-x}, \]
\[ y'(x) = A \cos x - B \sin x - C e^{-x}. \]

We get

\[ y' + y = (A + B) \cos x + (A - B) \sin x + (C - C)e^{-x}, \]

and the \( C \)'s cancel, thus we get the same solution for \( A, B \) as before. Our solution is then

\[ y(x) = \frac{1}{2} (\sin x - \cos x) + C e^{-x}. \]

Plugging in \( x = 0 \) gives

\[ y(0) = C - \frac{1}{2} = 4, \]

so \( C = 9/2 \), and the full solution is

\[ y(x) = \frac{1}{2} (\sin x - \cos x) + \frac{9}{2} e^{-x}. \]