Recall Taylor polynomials

- **nth-order Taylor term** of \( f(x) \) at \( a \):
  \[
  \frac{f^{(n)}(a)}{n!} (x - a)^n.
  \]

- **nth-order Taylor polynomial** of \( f(x) \) at \( a \):
  \[
  T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k
  \]
  
  \[
  = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 + \frac{1}{6} f'''(a)(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n.
  \]
Another way to think of this

Let us ask what is special about the function \( \frac{(x - a)^n}{n!} \) evaluated at \( a \).

Start with \( n = 1 \):

- Define \( p_1(x) = (x - a) \). Then:
  - \( p_1'(x) = 1 \), \( p_1''(x) = 0 \), \ldots
  - Now plug in \( a \). We have
    \[
    p_1(a) = 0, \quad p_1'(a) = 1, \quad p_1''(a) = 0, \quad p_1'''(a) = 0, \ldots
    \]
Define \( p_2(x) = \frac{1}{2}(x - a)^2 \).

Then
\[
p_2'(x) = (x - a), \quad p_2''(x) = 1, \quad p_2'''(x) = 0, \ldots
\]

Again plug in \( a \):
\[
p_2(a) = 0, \quad p_2'(a) = 0, \quad p_2''(a) = 1, \quad p_2'''(a) = 0, \quad p_2''''(a) = 0, \ldots
\]
Define $p_3(x) = \frac{1}{6}(x - a)^2$.

Then

$$p_3'(x) = \frac{1}{2}(x - a)^2, \quad p_3''(x) = (x - a), \quad p_3'''(x) = 1, \quad p_3''''(x) = 0$$

Again plug in $a$:

$$p_3(a) = 0, \quad p_3'(a) = 0, \quad p_3''(a) = 0, \quad p_3'''(a) = 1, \quad p_3''''(a) = 0, \ldots$$
Table!

<table>
<thead>
<tr>
<th></th>
<th>$x - a$</th>
<th>$\frac{1}{2}(x - a)^2$</th>
<th>$\frac{1}{6}(x - a)^3$</th>
<th>$\frac{1}{24}(x - a)^4$</th>
<th>$\frac{1}{120}(x - a)^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st deriv at $a$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2nd deriv at $a$</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3rd deriv at $a$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4th deriv at $a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>5th deriv at $a$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$$f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2} + f'''(a)\frac{(x - a)^3}{6} + \ldots$$

**Side note:** Looks like an identity matrix.
Analogy

\[ \vec{i} = (1, 0, 0), \quad \vec{j} = (0, 1, 0), \quad \vec{k} = (0, 0, 1), \]

For example,

\[ (2, 1, 3) = 2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1) = 2\vec{i} + 1\vec{j} + 3\vec{k}. \]
We will also use the “big O” notation as follows:

The term $O(x^p)$, then this means “any term of the size $x^p$ or smaller”

For example, near 0, $x^3$ is smaller than $x^2$ is smaller than $x$, etc.

Examples of expressions that are $O(x^2)$:

$$x^2, \quad x^2 + 2x^6, \quad x^3, \quad x^3 + x^5, \ldots$$

Expressions that are not:

$$1, \quad x, \quad 1 + x + x^2, \quad x + x^6$$

Example: a polynomial is $O(x^2)$ **only if** the constant and linear term are both zero.
An **asymptotic series** is a polynomial plus a big $O$ term:

- $1 + x + O(x^2)$
- $x - x^3/3 + O(x^5)$

For now, a **Taylor series** is a Taylor polynomial with a big $O$ term:

$$
\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)
$$

$$
\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)
$$

$$
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4)
$$

$$
\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + O(x^5)
$$

The last term is just keeping track of what we are not writing down.

We can also consider a Taylor series at a different point:

$$f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + O((x - a)^3)$$
When we say **3rd order Taylor series** we mean up to, and including, the 3rd order term:

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4).
\]

When we say **the first three terms in the Taylor series** we mean the first three nonzero terms:

\[
\sin(x) = x - \frac{x^3}{3} + \frac{x^5}{120} + O(x^7).
\]
To the board!
Let us try and rate the following statement for its reasonableness:

- “During the Olympics, I was watching Usain Bolt. I watched all his races. I saw how he set up in the blocks, watched his stride, everything.
- He makes running the 100 in under 10 seconds look easy.
- Then I went out to the track myself, and I couldn’t run the 100 in under 10 seconds.
- This surprises and discourages me.”
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Questions:

- How reasonable an expectation does this person have?
- What could we say to this person to give them a more realistic outlook, and encourage them?
Let us try and rate the following statement for its reasonableness:

- “Professor DeVille, I come to all of your lectures and watch carefully how you solve all the problems.
- You make solving these problems look easy.
- Then I go home to do the problems, and I get completely stuck.
- This surprises and discourages me.”