Math 231E, Lecture 7.
Limits using Taylor Series

1 Computing limits using Taylor series

Example 1. Let us now consider the limit

\[ \lim_{x \to 0} \frac{\sin(x)}{x}. \]

We cannot use the Limit Law, since the denominator goes to zero. We know that one way to do this is l’Hôpital’s Rule, but if we have Taylor series there is a better way to go.

Recall the Taylor series for \( \sin(x) \):

\[ \sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7), \]

so

\[ \frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6). \]

It is easy to see if we take the limit as \( x \to 0 \), the right-hand side goes to one, so

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1. \]

In fact, we can use Taylor series to derive l’Hôpital’s Rule, as follows: let us say that \( \lim_{x \to a} f(a) = \lim_{x \to a} g(x) = 0 \). Then we compute

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(a) + f'(a)(x-a) + O(x-a)^2}{g(a) + g'(a)(x-a) + O(x-a)^2}. \]

We know that \( f(a) = g(a) = 0 \), so

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a) + O(x-a)^2}{g'(a)(x-a) + O(x-a)^2} = \lim_{x \to a} \frac{f'(a)}{g'(a)} + O(x-a)^2, \]

where we got the last equality by dividing by \( (x-a) \). But we can then use the Limit Law (as long as \( g'(a) \neq 0 \)) and obtain

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}. \]

Example 2. We can do much more complicated examples using Taylor series. For example, say that we want to compute

\[ \lim_{x \to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)}. \]
Let us use Taylor series. We have

\[
\begin{align*}
\cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6), \\
\cos(x^2) &= 1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12}), \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(x^4), \\
e^{x^2} &= 1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16}), \\
\sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7), \\
\sin(x^4) &= x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28}).
\end{align*}
\]

So we have

\[
\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \left(1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right) - \left(1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16})\right) = x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28}).
\]

Adding like terms in the numerator gives

\[
\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}x^4 - \frac{11}{24}x^8 + O(x^{12}).
\]

We see that every term in the expression is divisible by \(x^4\), so divide this out to obtain

\[
\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2} - \frac{11}{24}x^4 + O(x^8) = 1 - \frac{x^6}{6} + \frac{x^{16}}{120} + O(x^{24}),
\]

and taking limits as \(x \to 0\) on both sides gives

\[
\lim_{x \to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}.
\]
2 Using Taylor Series for vertical asymptotes

Recall the calculation from last time:

\[
\frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = \frac{\left(1 - \frac{x^4}{2} + \frac{x^8}{24} + O(x^{12})\right) - \left(1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{6} + O(x^{16})\right)}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}
\]

\[
= \frac{-\frac{3}{2}x^4 - \frac{11}{24}x^8 + O(x^{12})}{x^4 - \frac{x^{12}}{6} + \frac{x^{20}}{120} + O(x^{28})}
\]

\[
= \frac{-\frac{3}{2} - \frac{11}{24}x^4 + O(x^8)}{1 - \frac{x^8}{6} + \frac{x^{16}}{120} + O(x^{24})},
\]

and so

\[
\lim_{x \to 0} \frac{\cos(x^2) - e^{x^4}}{\sin(x^4)} = -\frac{3}{2}.
\]

What worked out here was that the numerator and the denominator had the same leading order, so we were able to cancel terms and get a nice finite number.

What if this doesn’t happen? Let us consider the case

\[
\lim_{x \to 0} \frac{\cos(x)}{x^2} = \lim_{x \to 0} \frac{1 + \frac{x^2}{2} + O(x^4)}{x^2} = \frac{1}{x^2} + \frac{1}{2} + O(x^2).
\]

Now, we know

\[
\lim_{x \to 0} \frac{1}{x^2} = \infty,
\]

so \(\lim_{x \to 0} \frac{\cos(x)}{x^2} = \infty\) as well.

Moreover, this tells us more information that just that the limit is \(\infty\), it also tells us how quickly we approach \(\infty\); let us formalize this with a definition:

**Definition 1.**

- If \(\lim_{x \to a} f(x) = \infty\) and \(\lim_{x \to a} x^p f(x) = \infty\), then we say that \(f(x)\) goes to \(\infty\) faster than \(1/x^p\).

- If \(\lim_{x \to a} f(x) = \infty\) and \(\lim_{x \to a} x^p f(x) = 0\), then we say that \(f(x)\) goes to \(\infty\) slower than \(1/x^p\).

- If \(\lim_{x \to a} f(x) = \infty\) and \(\lim_{x \to a} x^p f(x) = L\), with \(0 < L < \infty\), then we say that \(f(x)\) goes to \(\infty\) as fast as \(1/x^p\), or, goes like \(1/x^p\).

We can see this from the Taylor expansions in a way that is difficult to see from other techniques. For example, let us consider the function

\[
f(x) = \frac{1}{e^{x^4} - 1}.
\]

It is clear that this function should have some sort of singularity at \(x = 0\), since the numerator is one and the denominator is zero. To be more precise:

\[
\frac{1}{e^{x^4} - 1} = \frac{1}{\left(1 + x^4 + \frac{x^8}{2} + O(x^{12})\right) - 1} = \frac{1}{x^4 + x^8/2 + O(x^{12})} = \frac{1}{x^4} \cdot \frac{1}{1 + x^4/2 + O(x^8)}.
\]
Now, it is clear that
\[
\lim_{x \to 0} \frac{1}{1 + x^4 / 2 + O(x^8)} = \frac{1}{1} = 1,
\]
so
\[
\lim_{x \to 0} x^4 \cdot \frac{1}{e^{x^4} - 1} = 1.
\]
Therefore, according to our definition, we say that \( f(x) \) goes to \( \infty \) like \( x^{-4} \).