1 Limits at $\infty$ and horizontal asymptotes

We now would like to define limits as $x \to \infty$, and we can do so in the following manner:

We say that

\[
\lim_{x \to \infty} f(x) = L, \text{ if } \lim_{x \to 0^+} f \left( \frac{1}{x} \right) = L.
\]

In short, we first plug in $1/x$ for $x$, then take the limit $x \to 0$ from the right.

**Example 1.** We want to compute

\[
\lim_{x \to \infty} \frac{3x^2 + 2x - 7}{x^2 - 4x + 1}.
\]

We plug in $1/x$ for $x$ to obtain

\[
\frac{3(1/x)^2 + 2(1/x) - 7}{(1/x)^2 - 4(1/x) + 1} = \frac{3}{x^2} + \frac{2}{x} - \frac{7}{1/x^2} \cdot \frac{1}{x^2} - \frac{4}{1/x} + \frac{1}{1} = \frac{3}{x^2} - \frac{7}{x^2} \cdot \frac{1}{x^2} - \frac{4}{x} + \frac{1}{1}.
\]

and taking the limit $x \to 0^+$, we obtain 3.

We also say that

\[
\lim_{x \to -\infty} f(x) = L, \text{ if } \lim_{x \to 0^-} f \left( \frac{1}{x} \right) = L.
\]

**Example 2.** Let us compute

\[
\lim_{x \to \infty} \frac{e^x}{x^n}.
\]

for some fixed $n > 0$. We know that $e^x > x^{n+1}/(n+1)!$. (Why is this?) Then we have

\[
\frac{e^x}{x^n} > \frac{x^{n+1}}{x^n(n+1)!} = \frac{x}{(n+1)!}.
\]

Now,

\[
\lim_{x \to \infty} \frac{x}{(n+1)!} = \lim_{x \to 0^+} \frac{1/x}{(n+1)!} = \frac{1}{(n+1)!} \lim_{x \to 0^+} \frac{1}{x} = \infty.
\]

Therefore

\[
\lim_{x \to \infty} \frac{e^x}{x^n} = \infty.
\]
Example 3. Similarly, we would like to compute

$$\lim_{x \to \infty} x^n e^{-x}.$$ 

From the arguments above, we have that

$$\frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$ 

Also, for $x > 0$, we have $x^n e^{-x} > 0$, so

$$0 < \frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$ 

We compute

$$\lim_{x \to \infty} \frac{(n+1)!}{x} = \lim_{x \to 0^+} \frac{(n+1)!}{1/x} = (n+1)! \lim_{x \to 0^+} x = 0,$$

so by the Squeeze Theorem we have

$$\lim_{x \to \infty} x^n e^{-x} = 0.$$ 

This is why we would say that “$e^{-x}$ decays to zero faster than any polynomial as $x \to \infty$”.

2 Squeeze Theorem

Theorem 1. If $f(x) \leq g(x) \leq h(x)$ for $x$ near $a$, and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} g(x) = L$.

Example 4. We compute

$$\lim_{x \to 0} x^2 \sin \left(\frac{1}{x}\right).$$

We cannot use a Limit Law, since $\lim_{x \to 0} \sin(1/x)$ does not exist. However, let us Squeeze it:

$$-1 < \sin(1/x) < 1$$

and we know

$$\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0,$$

so

$$\lim_{x \to 0} x^2 \sin(1/x) = 0.$$ 

See Figure 1.

3 Rule of Thumb

When considering rational functions\(^1\) with limits at infinity, there is a good rule of thumb to follow. We’ll work through three examples of this first.

\(^1\)A rational function is a quotient of two polynomials
Example 5. Consider the limit
\[
\lim_{x \to \infty} \frac{3x^3 + 2x - 7}{4x^3 + x^2 - x - 1}.
\]
Plug in \(1/x\) and take the limit as \(x \to 0^+\):
\[
\lim_{x \to 0^+} \frac{3x^3 + 2x - 7}{4x^3 + x^2 - x - 1} = \lim_{x \to 0^+} \frac{1/x^3 \cdot (3 + 2x^2 - 7x^3)}{4 + x - x^2 - x^3} = \lim_{x \to 0^+} \frac{3 + 2x^2 - 7x^3}{4 + x - x^2 - x^3} = \frac{3}{4}.
\]

**Rule 1:** if the top and bottom are the same degree, then the limit is the ratio of the leading coefficients. In this example, we see that both top and bottom are exactly cubic, and the leading coefficients are 3 and 4.

Example 6. Now consider
\[
\lim_{x \to \infty} \frac{-3x^3 + 2x - 7}{x^2 - x - 1}.
\]
Plug in \(1/x\) and take the limit as \(x \to 0^+\):
\[
\lim_{x \to 0^+} \frac{-3x^3 + 2x - 7}{x^2 - x - 1} = \lim_{x \to 0^+} \frac{1/x^3 \cdot (-3 + 2x^2 - 7x^3)}{x - x^2 - x^3} = \lim_{x \to 0^+} \frac{-3 + 2x^2 - 7x^3}{x - x^2 - x^3}.
\]
We see here that the numerator goes to \(-3\) and the denominator to \(0^+\), so the limit is \(-\infty\).

**Rule 2:** If the numerator is of a higher degree than the denominator, then the limit is \(\pm \infty\), where the sign is the same as the sign of the leading-order coefficient of the numerator.

Example 7. Consider the limit
\[
\lim_{x \to \infty} \frac{2x - 7}{4x^3 + x^2 - x - 1}.
\]
Plug in \(1/x\) and take the limit as \(x \to 0^+\):
\[
\lim_{x \to 0^+} \frac{2x - 7}{4x^3 + x^2 - x - 1} = \lim_{x \to 0^+} \frac{1/x^3 \cdot (2x^2 - 7x^3)}{4 + x - x^2 - x^3} = \lim_{x \to 0^+} \frac{2x^2 - 7x^3}{4 + x - x^2 - x^3}.
\]
We see here that the numerator goes to 0 while the denominator goes to 4, so the limit is zero.

**Rule 3:** if the denominator is of a higher degree than the numerator, then the limit is zero.

**Exercise:** Which of these rules change, and how, if we take the limit \( x \to -\infty \)?