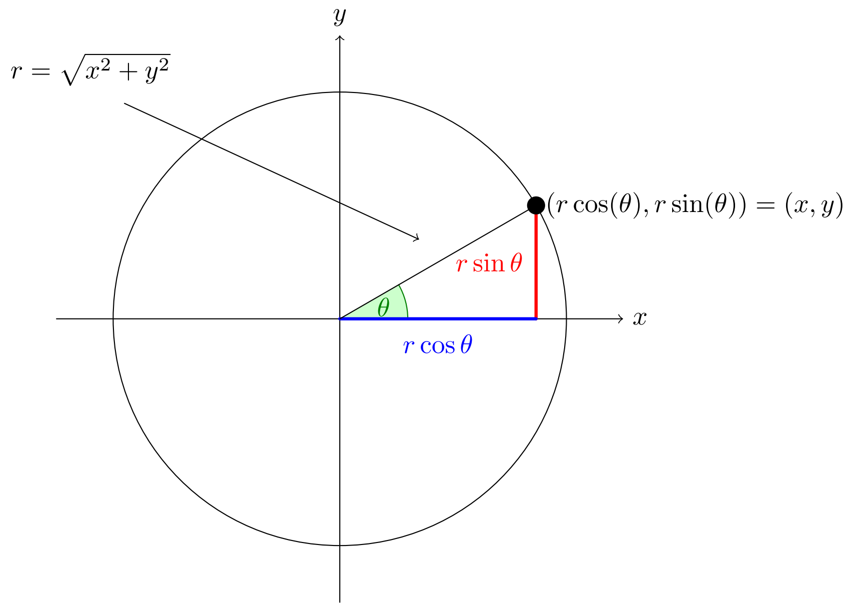
Math 231E, Lecture 34.
Polar Coordinates and Polar Parametric Equations

1 Definition of polar coordinates

Let us first recall the definition of Cartesian coordinates: to each point in the plane we can associate a pair of numbers (x, y) as the **Cartesian** or **rectangular** coordinates of this point.

Note that Cartesian coordinates have the property that each pair of numbers corresponds to a unique point, and vice versa.

Now, for each point in the plane, we can define the **polar coordinates** of the point as the pair (r, θ), where r is the distance of the point from the origin, and θ is the angle between the line segment from the origin to the point and the x-axis.



We see from the picture that we have conversion formulas from polar to rectangular coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and conversion formulas the other way:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x).$$

One thing to note about polar coordinates is that there is not a one-to-one correspondence between points in the plane and polar coordinates. For example, the point $(x, y) = (1, 1)$ can be represented by the polar coordinates $(r, \theta) = (\sqrt{2}, \pi/4)$, but it can also be represented by the polar coordinates $(r, \theta) = (\sqrt{2}, 9\pi/4)$, etc.

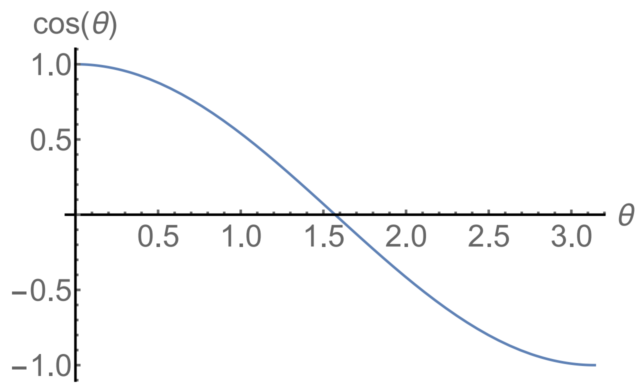
We will also use a (perhaps strange) convention that allows for negative r, as follows: if $r < 0$, then the polar coordinates (r, θ) are defined to be $(|r|, \theta + \pi)$.

2 Polar graphs

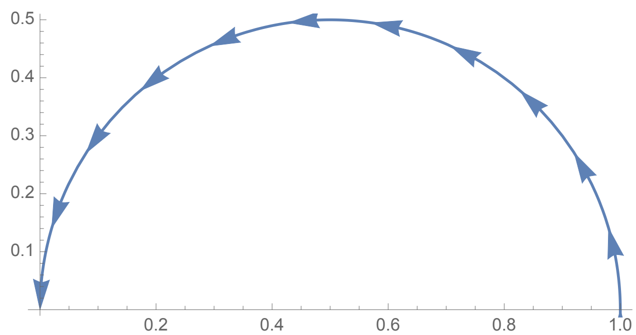
We will find it useful to "graph" the expression $f(r, \theta) = 0$, which is just a picture of the set of points in the plane that satisfy this equation. In most cases, this equation is written in "explicit" form as $r = g(\theta)$, but not always. When it is done this way, we might specify a domain for θ as well.

Example 2.1. The simplest examples are the cases where there is only one variable. We can consider the equation $r = 2$, which is the circle of radius 2, and $\theta = \pi$, which is the negative x-axis.

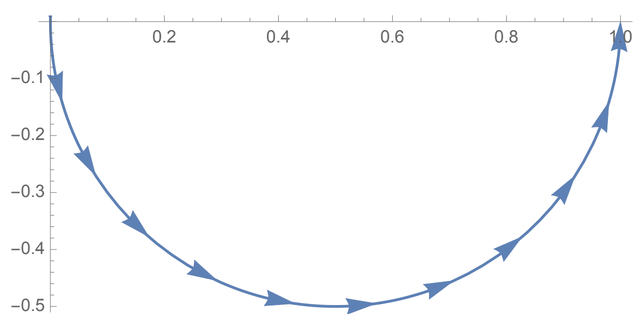
Example 2.2. Now consider $r = \cos\theta$, $\theta \in [0, 2\pi]$. If we plot the graph of $\cos\theta$ versus θ, we obtain:



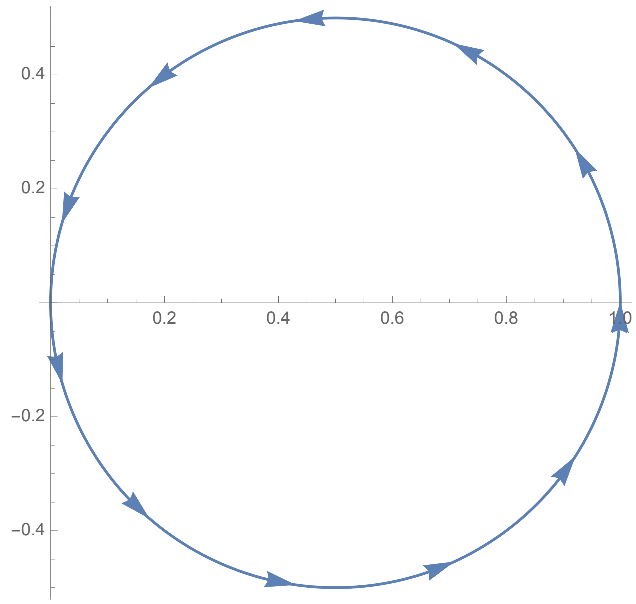
Note the first half of the graph, with domain $\theta \in [0, \pi/2]$. This function decreases from one to zero. If we plot just this in polar form, we obtain



Now notice the next half of the curve, with $\theta \in [\pi/2, \pi]$. In this case, we have a negative r, which decreases from zero to minus one. But recall that having a negative r means we flip the angle by π, so while the angle is in the top left quadrant, this means we are plotting in the bottom right quadrant, moving from zero to one:



Putting these together gives us an entire circle:



To check that this makes sense in rectangular coordinates, let convert the original equation. We have

$$r = \cos\theta,$$

and multiplying both sides of this equation by r gives

$$r^2 = r\cos\theta,$$

or

$$x^2 + y^2 = x.$$

Writing

$$x^2 - x = x^2 - x + \frac{1}{4} - \frac{1}{4} = (x - 1/2)^2 - \frac{1}{4},$$

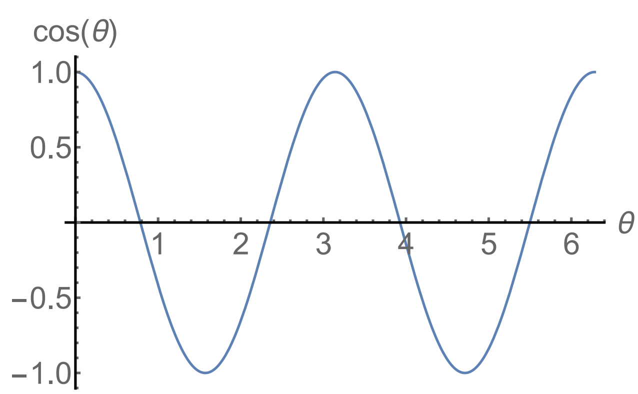
we finally obtain the equation

$$(x - 1/2)^2 + y^2 = \frac{1}{4},$$

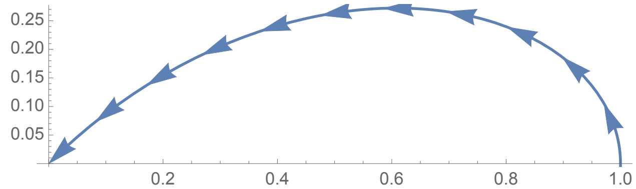
which is the (rectangular) equation for a circle of radius $1/2$ centered at $(1/2, 0)$.

Example 2.3. Let's change this up, and use the equation

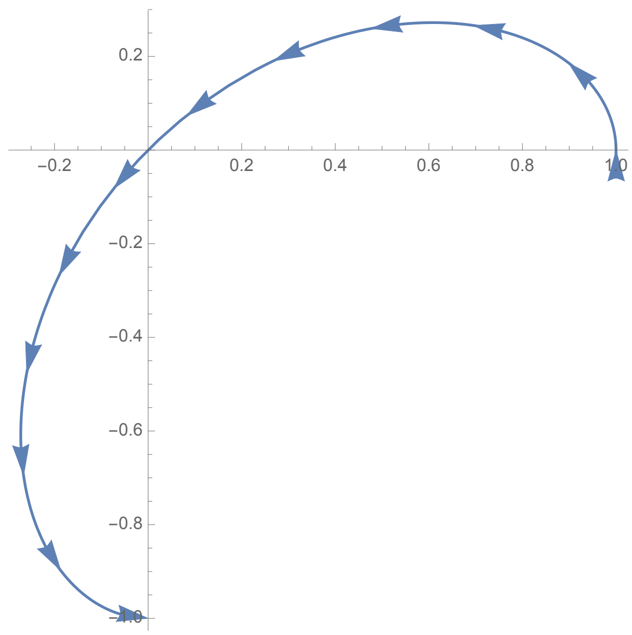
$$r = \cos 2\theta.$$



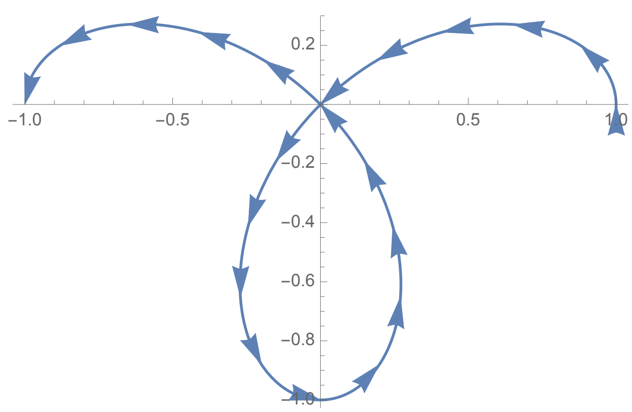
Consider the first eighth of the graph with $\theta \in [0, \pi/4]$. Here we start at one and decrease to zero, so the polar plot is



Notice that the curve is coming into the origin when $\theta = \pi/4$, so with slope 1. Now the curve becomes negative, and we are thus plotting on "the other side". By the time we get to $\theta = \pi/2$, we have $r = -1$, and this is the same as $r = 1, \theta = 3\pi/2$:



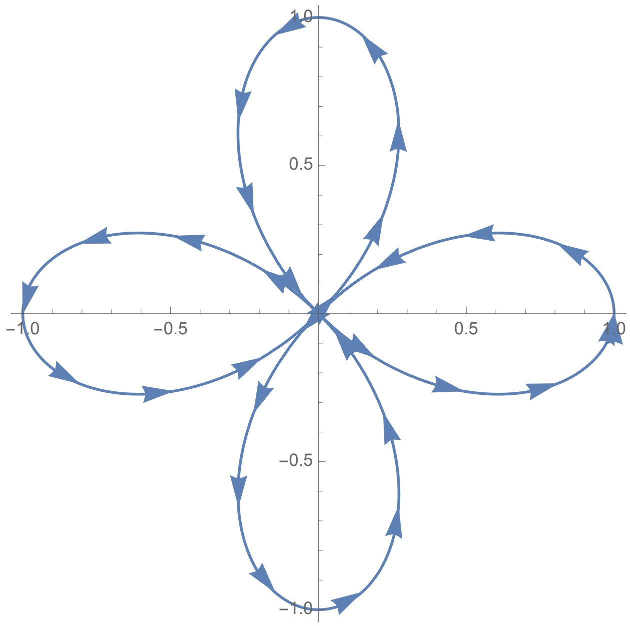
If we consider the domain $\theta \in [\pi/2, \pi]$, then notice that $\cos\theta$ moves from -1 to 1, so we will end up at $(1, \pi)$ after passing through $(0, 3\pi/2)$:



Finally, let us note that

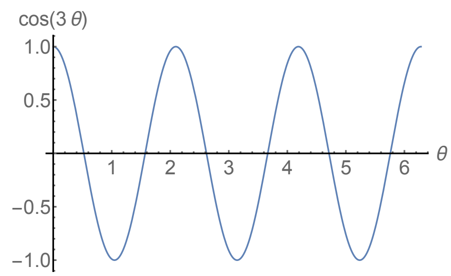
$$\cos(2(\theta + \pi)) = \cos(2\theta),$$

so the remainder of the plot is just the original plot rotated by π, giving:

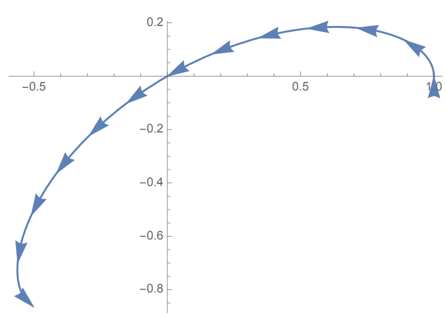


Four leaf clover!!

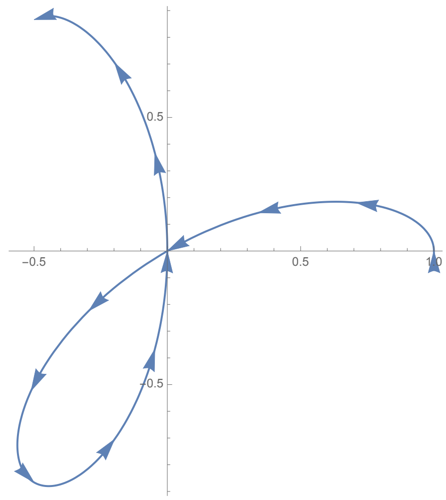
Example 2.4. Let us now consider $r = \cos 3\theta$.



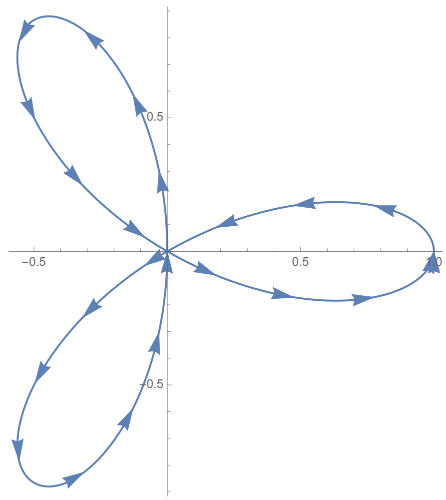
Notice that this function goes from 1 to -1 as θ goes from 0 to $\pi/3$. Recall that the polar coordinates $(-1, \pi/3)$ also correspond to $(1, \pi/3 + \pi)$, so we have the curve:



The function $\cos 3\theta$ then goes back to 1 at $\theta = 2\pi/3$, so we obtain:



And then when θ has reached π, we again have $r = -1$. But $(-1, \pi)$ is the same point as $(1, 0)$, so we have



And now we've arrived back where we started! But notice that

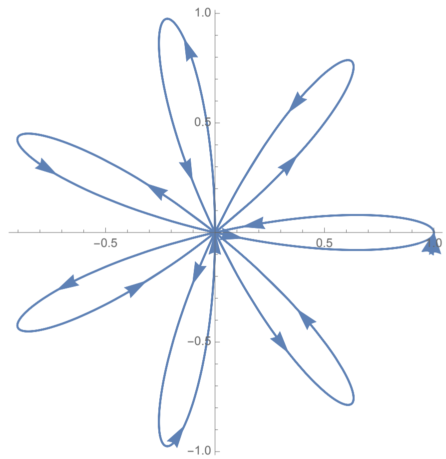
$$\cos(3(\theta + \pi)) = \cos(3\theta + 3\pi) = \cos(3\theta + \pi) = -\cos(3\theta).$$

This means that the second half of the θ domain will just trace the curve over again. So in this case, we obtain a three-leaf clover, but it is covered twice!

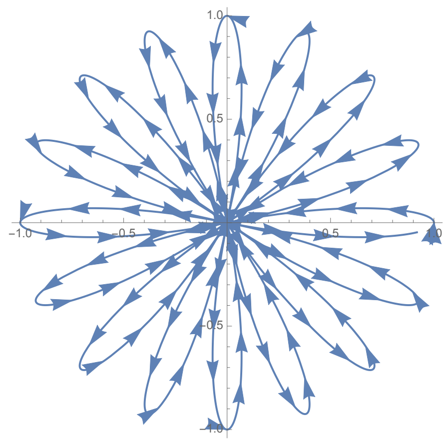
Example 2.5. It is probably not so surprising, given this, that the plot of

$$r = \cos(n\theta)$$

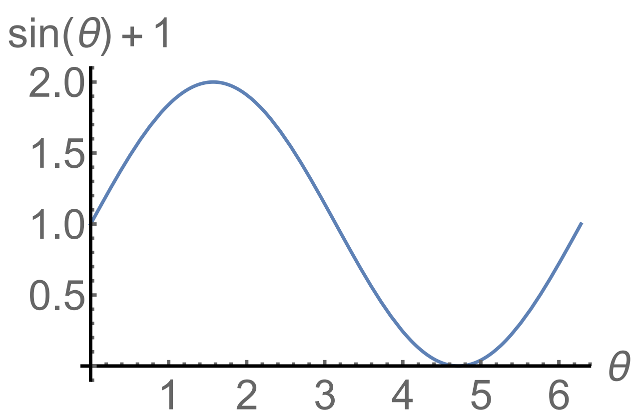
is a $2n$-leaf clover when n is even, and an n-leaf clover, covered twice, when n is odd. For example, if $n = 7$ we obtain 7 leaves:



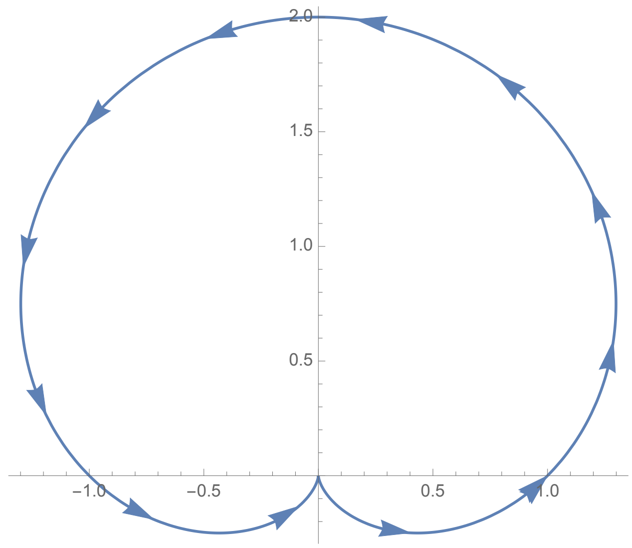
and when $n = 8$ we obtain 16 leaves:



Example 2.6. Another example is the **cardioid** $r = 1 + \sin\theta$:



We have $r \geq 0$ so we are never "crossing through the origin". We increase to a maximum of 2 at $\theta = \pi/2$, and then down to a minimum of 0 at $\theta = 3\pi/2$:



2.0
1.5
1.0
0.5
−1.0
−0.5
0.5
1.0