Math 231E, Lecture 33.
Parametric Calculus

1 Derivatives

1.1 First derivative

Now, let us say that we want the slope at a point on a parametric curve. Recall the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

which exists as long as $dx/dt \neq 0$.

Example 1.1. Reconsider the circle example above. We have

$$\frac{dx}{dt} = -\sin(t), \quad \frac{dy}{dt} = \cos(t),$$

so

$$\frac{dy}{dx} = \frac{\cos(t)}{-\sin(t)} = -\frac{x}{y}.$$

What kind of tangency do we have on a parametric curve?

1. If $dy/dt = 0$ and $dx/dt \neq 0$, then the curve should have a **horizontal tangency**.

2. If $dx/dt = 0$ and $dy/dt \neq 0$, then the curve should have a **vertical tangency**.

3. If both derivatives are zero, then all kinds of things can happen: we can have vertical or horizontal tangencies, we can have a "cusp", we can even have a regular point that looks perfectly fine.

Although we don't say much about it here (you will see much more in MATH 241!), we can also think of the **tangent vector** to a curve as the vector:

$$\left(\frac{dx}{dt}, \frac{dy}{dt} \right)$$

Its slope gives dy/dx in the computations above. Note that it also has another parameter associated to it, its length:

$$\sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2},$$

which we can think of as a "speed" at which we traverse the curve. (More on this below.)

1.2 Second (and higher) derivatives

What if we are interesting in computing where a parametric curve is concave up or down, for example? Here we would need the second derivative d^2y/dx^2. How do we compute this?

Recall the chain rule. If ☺ is a function of x and x is a function of t, then

$$\frac{d☺}{dt} = \frac{d☺}{dx}\frac{dx}{dt}, \tag{1}$$

or

$$\frac{d☺}{dx} = \frac{\dfrac{d☺}{dt}}{\dfrac{dx}{dt}}.$$

This is the formula we used above, but with ☺ = y. To get the second derivative d^2y/dx^2, let us choose ☺ = dy/dx, giving

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d(dy/dx)}{dt}}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\left(\dfrac{dy/dt}{dx/dt}\right)}{\dfrac{dx}{dt}}$$

It is a complicated formula. I wouldn't suggest memorizing it, but do recall how to derive it, because this and related formulas are useful!

2 Cycloid, revisited

Recall the equations for the cycloid:

$$x = r(t - \sin t), \quad y = r(1 - \cos t). \tag{2}$$

Let us first compute the derivative at each point on the cycloid. Note that we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{r\sin t}{r(1 - \cos t)} = \frac{\sin t}{1 - \cos t}.$$

We see that the slope is zero when $\sin t = 0$ and $1 - \cos t \neq 0$, which occurs at every odd multiple of π. Notice that whenever the denominator is zero, so is the numerator, so we have to be clever to compute the derivative. For example, let us consider $t = 0$. We formally have a $0/0$ indeterminate form here. Expanding in Taylor series at $t = 0$, we obtain

$$\frac{dy}{dx} = \frac{t - t^3/6 + O(t^5)}{t^2/2 + O(t^4)} = \frac{1}{2t} + O(t).$$

We see that the left and right limits are different! So in fact we have

$$\lim_{t \to 0+}\frac{dy}{dx} = \infty, \quad \lim_{t \to 0-}\frac{dy}{dx} = -\infty.$$

This makes sense from the picture: the slope is pointing down as we approach each cusp from the left, and pointing up as we approach each cusp from the right.
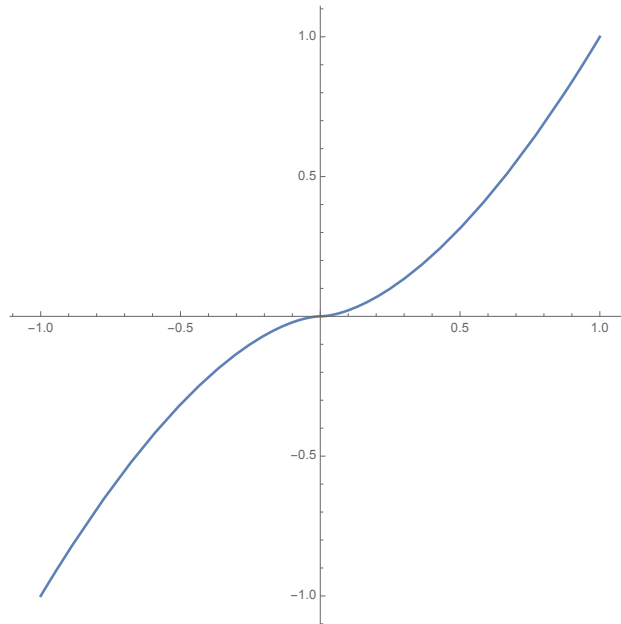
3   Other examples with indeterminate derivatives

We have already seen from the cycloid that a point with an indeterminate derivative can have a vertical tangency in a cusp-like manner.

We can also see that a horizontal and vertical tangencies are possible. For example, consider the parametric curve

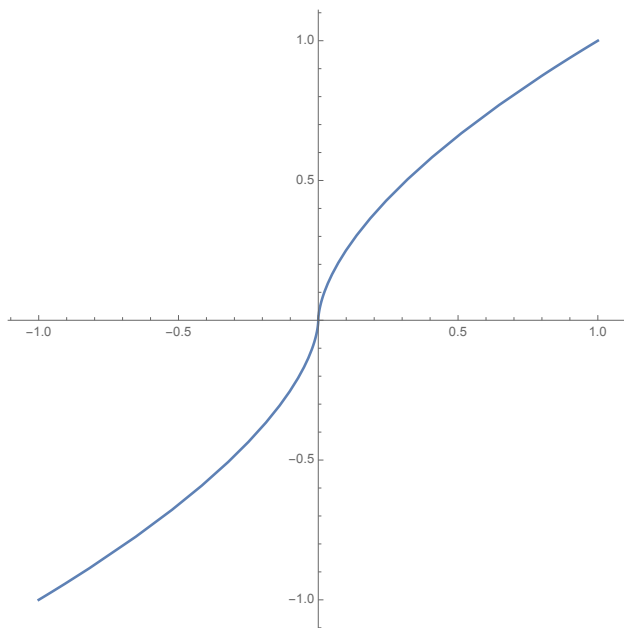$$x = t^3, \quad y = t^5, \quad t \in [-1, 1].$$

We plot this:

Notice that $x'(t) = 3t^2$, so $x'(0) = 0$, and $y'(t) = 5t^4$, so $y'(0) = 0$. We can see from the picture that the curve seems to have a horizontal tangency. In fact, note that x, y satisfy $y = x^{5/3}$, and therefore

$$\frac{dy}{dx} = \frac{5}{3}x^{2/3},$$

so at $x = 0$ this slope is, in fact, zero.

We can also obtain a vertical tangency by flipping the role of x, y in the previous example:

$$x = t^5, \quad y = t^3, \quad t \in [-1, 1].$$

3

Also, note that we can actually obtain any slope we want at an indeterminate point. For example, consider

$$x = t^3, \quad y = \alpha t^3.$$

Note that at $t = 0$, both of these functions have a zero derivative. But we can see that $y = \alpha x$, so the curve is just a line with slope α. We can choose α to be any number we want to get any slope we want.

Basically, when both functions have a zero derivative, anything can happen!

4 Area

We might now try to compute the area underneath a parametric curve, or more specifically, the (signed) area between a curve and the x-axis. Now, if we have a curve $y = f(x)$ where $x \in [a, b]$, then the area under the curve is given by

$$\int_a^b f(x)\, dx = \int_a^b y\, dx.$$

Now consider the parametric curve

$$x = f(t), \quad y = g(t), \quad t \in [c, d],$$

where $f(c) = a$ and $f(d) = b$. Writing $dx = f'(t)\, dt$, we then obtain

$$\int_a^b y\, dx = \int_c^d g(t) f'(t)\, dt.$$

In the case where we have $x = t, y = g(t)$, notice that this recovers the standard integral, but this allows us to consider the general parametric case.

For example, what is the area under one leaf of the cycloid? We have

$$\int_0^{2\pi} r(1 - \cos t) \cdot r(1 - \cos t)\, dt$$

$$= \int_0^{2\pi} r^2(1 - \cos t)^2\, dt$$

$$= r^2 \int_0^{2\pi} 1 - 2\cos t + \cos^2 t\, dt$$

$$= r^2 \int_0^{2\pi} 1 - 2\cos t + \frac{1}{2}(1 + \cos 2t)\, dt.$$

The two trig functions will have a zero average, so the final answer we obtain here is

$$r^2 \int_0^{2\pi} \frac{3}{2}\, dt = 3\pi r^2.$$

5 Arc Length

Recall our derivation of arc length earlier in the semester. If we break up a curve into many small line segments, notice that we have

$$ds^2 = dx^2 + dy^2,$$

and thus our integrand is

$$\int_c^d \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$$

For example, if we want the length of one leaf of the cycloid, we compute

$$\int_0^{2\pi} \sqrt{r^2(1 - \cos t)^2 + r^2 \sin^2 t} = r \int_0^{2\pi} \sqrt{2(1 - \cos t)}\, dt.$$

To do this integral, let us recall the trig formula

$$\cos^2 t = \frac{1}{2}(1 - \cos 2t).$$

Solving gives

$$\cos 2t = 2\cos^2 t - 1 = 1 - 2\sin^2 t.$$

This means we can also write

$$1 - \cos t = 1 - (1 - 2\sin^2(t/2)) = 2\sin^2(t/2).$$

So

$$\sqrt{2(1 - \cos t)} = \sqrt{4\sin^2(t/2)} = 2\sin(t/2).$$

Then our integral is

$$2r \int_0^{2\pi} \sin(t/2)\, dt = -4r \cos(t/2) \Big|_{t=0}^{t=2\pi} = -4r(-1 - 1) = 8r.$$

We can also consider the arc length of a circle, which is a much simpler calculation. A circle of radius one is parameterized by

$$x = \cos t, \quad y = \sin t, t \in [0, 2\pi].$$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} dt = 2\pi.$$

Of course we already knew that.