Taylor Polynomials

The Taylor series \( f(x) \) is very often hard to compute and so we use the Taylor polynomial:

\[
T_N(x) = \sum_{n=0}^{N} a_n x^n = a_0 + a_1 x + \ldots + a_N x^N
\]

We put \( R_N(x) = f(x) - T_N(x) = \sum_{n=N+1}^{\infty} a_n x^n \).

1. Based on your notes, what is \( R_N(x) \) equal to?

We use \( R_N \) to approximate the Taylor series using Taylor polynomials.

2. a) Write down the Taylor polynomial of degree 5 for \( e^x \) at 0 (should take no work).

b) Use Taylor’s Theorem to find the maximum error in the approximation \( e^x \approx T_5(x) \) on the interval \([0, 1/2]\).

3. a) Find the Taylor polynomial of degree 2 for \( f(x) = x^{3/2} \) at 4

b) Use Taylor’s Theorem to find the maximum error in the approximation \( f(x) \approx T_2(x) \) on the interval \([4, 4.5]\).
We can use Taylor polynomials to find some limits very fast.

4. Find the following limits:
   a) \( \lim_{x \to 0} \frac{\sin(x)}{x} \) (use the degree 1 approximation)

   b) \( \lim_{x \to \infty} \frac{1 - \cos(x)}{x^2} \) (use the degree 2 approximation)

5. Show that the degree 1 Taylor polynomial for \((1 + x)^K\) is \(T_1(x) = 1 + Kx\) (you can just write down the first two terms of the binomial series).

So we have \((1 + x)^K \approx 1 + Kx\) if \(x\) is small.
6. Pendulums have been used for centuries to keep time. Pendulums exhibit the property of *isochronism* when they swing through small angles. This means that the period of each swing (i.e. the amount of time which each swing takes) does not change very much as the angle of the swing changes. For small angles, the period is given by the formula

\[ T \approx 2\pi \sqrt{\frac{L}{g}} \]  

where \( L \) is the length of the pendulum and \( g \) is the gravitational constant.

However, the period of the pendulum *must* depend somehow on the size of the angle through which it swings (to convince yourself of this, imagine a pendulum which is thousands of feet long swinging through small angles, and then swinging through large angles).

Suppose that the maximum angle which the pendulum makes with vertical is \( \theta_{\text{max}} \), and set \( k = \sin\left(\frac{1}{2}\theta_{\text{max}}\right) \). Then the precise formula for the period of a pendulum is

\[ T = 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}. \]  

We will use series to reconcile equations (1) and (2).

a) Use Problem 5 to find the first two terms of the expansion of \( \frac{1}{\sqrt{1 - k^2 \sin^2 x}} \). (The answer involves \( k \) and \( \sin x \).)

b) Use part (a) and (2) to find the first two terms of the expansion of \( T \) in terms of \( k \) (the answer involves \( k \) only). Use the fact that \( \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4} \).

c) Explain very briefly why the approximation (1) is accurate when \( k \) is small.
We can also use the Taylor polynomials to find good approximations for series:

7. How can you use Taylor polynomials to approximate \( \ln(\frac{3}{2}) \) up to \( > 0.01 \)?

8. In Calculus I, we learned the concept of linear approximation. State the definition of a linear approximation and compare it to Taylor polynomials. What is the connection between them?

Remember we used linear approximations to approximate irrational numbers. The previous question tells us the Taylor polynomial gives us a better approximation. For example:

9. Use degree 3 Taylor polynomial to find an approximate value for \( \sqrt{10} \)