7.1: Integration by Parts

The Integration by Parts formula is a process of trading a difficult integral for an easier integral. If you understand how to derive the formula, you should never forget it. So, let’s start with that:

1. Prove the formula for Integration by Parts by completing the following steps:

   (a) Fill in the blank: \( \frac{d}{dx} f(x)g(x) = \) ____________________________.

   (b) Integrate both sides of the equation in part (a) with respect to \( x \) and use the Fundamental Theorem of Calculus to simplify.

   (c) Rewrite the equation from part (b) by solving for \( \int f(x)g'(x) \, dx \).

   (d) Parts (a)-(c) demonstrate that Integration by Parts is the opposite of what well-known rule for differentiation?

   (e) Rewrite the formula for integration by parts using \( u = f(x) \) and \( v = g(x) \).

   (f) Look back over what you just did. Congratulations, you just proved the integration by parts formula. Now you don’t have to memorize it!

Now, with the formula at hand solve the following example (just to make sure):

2. \( \int \ln(x) \, dx \)
The trickiest thing about integration by parts is how to determine $u$ and $v$ correctly. Unfortunately, there is no absolute way to answer this question and the best way to determine this is to solve many, many examples and gain experience. There is, however, a general guide which helps you most of the times. Most functions are built out of these five building blocks: logarithmic (L), inverse trigonometric (I), algebraic (A), trigonometric (T), and exponential (E). You can build the following mnemonic:

**LIATE**

where the rule is: *Pick $u$ in the order of LIATE.*

Now, with LIATE at hand let’s solve some problems:

3. Evaluate using integration by parts

   (a) $\int \arctan x \, dx$

   (b) $\int \frac{\ln x}{x^2} \, dx$

   (c) $\int e^{2x} x^2 \, dx$
4. (a) Integrate by parts to get a formula for $\int \sin^2 x \, dx$ which involves $\int \cos^2 x \, dx$.

(b) Evaluate $\int \sin^2 x \, dx$, by using part (a) and the identity $\cos^2 x = 1 - \sin^2 x$.

5. Sometimes you have to combine substitution and integration by parts:
   (a) $\int e^{\sqrt{x}} dx$

   (b) $\int t^5 e^{t^2} \, dt$

   (c) $\int x(2 + x^2) \ln(2 + x^2) \, dx$
As we saw in problem 3 (c) sometimes you have to use Integration by Parts formula several times. Let’s see another example:

6. We want to find $\int (\ln(x))^n \, dx$
(a) Find $\int (\ln(x))^2 \, dx$

(b) Find $\int (\ln(x))^3 \, dx$

(c) Now make an educated guess for $\int (\ln(x))^n \, dx$ and prove it.

One of the main applications of integration by parts is Fourier analysis, which has many applications in physics and engineering, in particular understanding waves and signals. The next question is an example of an integral which shows up all the time:

7. Show that $\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = 0$