A power series is a function of the form

\[ f(x) = \sum_{n=1}^{\infty} a_n x^n \]

Every power series has a radius of convergence \( r \) which determines where the function is defined. The first goal is to use power series to approximate \( \int_0^1 x \arctan x \, dx \).

1. For each function, write down the power series and the radius of convergence. You may express the radius of convergence by writing \( |x| < R \).

Step 1: \( \frac{1}{1-x} \)

Step 2: \( \frac{1}{1+x^2} \)

Step 3: \( \arctan x \)

Step 4: \( x \arctan x \)

Step 5: \( \int x \arctan x \, dx \)
2. Write down a series for \( \int_0^1 x \arctan x \, dx \).

So, whenever we cannot calculate the integral we should replace it with a power series and then take the integral to approximate the answer.

So let’s learn more about power series:

3. For which \( x \) does the following power series converge:
   a) \( \sum_{n=1}^{\infty} n^n x^n \)
   b) \( \sum_{n=1}^{\infty} (-1)^n \frac{x}{n^3 + 3} \)

4. You are given that \( \sum c_n (-3)^n \) converges, and that \( \sum c_n 5^n \) diverges.
   a) What are the possible values of the radius of convergence of the power series \( \sum c_n x^n \)?

   What can you say about the convergence/divergence of the following series?
   b) \( \sum c_n (-6)^n \)
   c) \( \sum c_n 2^n \)
   d) \( \sum c_n 4^n \)
   e) \( \sum c_n (-5)^n \)
5. For these series, use either the ratio or the root test to determine the radius of convergence. Then determine the interval of convergence.

a) \[ \sum_{n=1}^{\infty} (-1)^n \frac{(x - 3)^n}{n \cdot 5^n} \]

b) \[ \sum_{n=1}^{\infty} \frac{x^n}{e^{n\pi}} \]

c) \[ \sum_{n=1}^{\infty} \frac{n(x + 3)^n}{4^n} \]

d) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} (x + 2)^n \]
6. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge, but to what number do the partial sums converge?

a) Write out an expansion in powers of $x$ for $\frac{1}{1+x}$ (again).

b) Integrate both sides of the expression you found above (we’ll see later this is allowed) to get the power series expansion for $\ln(1+x)$. Now use this expansion to conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$.

SO, $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$. However, if we rearrange the terms in this series to $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$, it converges to $\frac{3}{2} \ln 2$ instead.

The MORAL: We can’t necessarily rearrange the terms in a series! There are other, even crazier rearrangements that converge to 0 or even diverge to $\infty$.

7. Sometime we can use series to find limits:

a) $\lim_{x \to 0} \frac{\sin(3x^2)}{x^2}$

b) $\lim_{x \to 0} \frac{1 + x - e^x}{x^2}$