11.1 Sequences

- A sequence \( \{a_n\} \) is **bounded above** if there exists an \( M \) with \( a_n \leq M \) for all \( n \) and **bounded below** if there exists an \( N \) with \( a_n \geq N \) for all \( n \).
- A sequence is \( \{a_n\} \) is **increasing** if \( a_n \leq a_{n+1} \) for all \( n \) and **decreasing** if \( a_n \geq a_{n+1} \) for all \( n \).

1. In the following determine whether each of the following sequences is

- bounded above/bounded below/neither
- increasing/decreasing/neither
- convergent. If the sequence converges give the limit.

a) \( a_n = 1 - 1/n^3 \) \( \{a_n\} = 0, \frac{7}{8}, \frac{26}{27}, \frac{255}{256}, \ldots \) 

b) \( b_n = 2 - (-1)^n \)

c) \( c_n = ne^{-n} \) \( \{c_n\} = 0.368, 0.271, 0.149, 0.0733, 0.0337, 0.0148, 0.0064, \ldots \) 

d) \( d_n = n2^n \)

e) \( e_n = \frac{n!}{2^n} \)
f) \( f_n = \frac{\sin(n)}{3^n} \)

g) \( g_n = n^{\frac{1}{n}} \)

2. Recall the fundamental geometric series

\[ 1 + r + r^2 + r^3 + \cdots = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{Divergent} & \text{if } |r| \geq 1 \end{cases} \]

Write each of the following series in the form \( a(1 + r + r^2 + r^3 + \ldots) \). Identify the value of \( r \) in each case. Find the sum of the series, or write “Diverges”.

a) \( \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots \)

b) \( \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots + \frac{1}{768} + \ldots \)

c) \( \sum_{n=1}^{\infty} 5(-2)^{n-1} \)

d) \( \sum_{n=2}^{\infty} \frac{22n-1}{7^n} \)
Remember that the partial sum is $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ and so $a_n = S_{n+1} - S_n$.

3. Given the partial sum $S_n = \frac{n}{n+1}$, find $a_n$ and $\sum_{n=1}^{\infty} a_n$.

4. Find $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$, by first finding the partial sum and then using methods similar to the previous problem.

5. Generalize the previous question: Find $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ for some sequence $\{a_n\}$ if you know that $a_1 = 1$, by finding partial sums.

6. Use the result from the previous problem find $\sum_{n=1}^{\infty} \ln(\frac{n}{n+1})$. 
7. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences converging to \( a \) and \( b \), respectively. What can we say about:
   a) The limit of \( \{a_n + b_n\} \)
   b) The limit of \( \{a_n b_n\} \)
   c) Using part a) and b) what can we say about \( \{a_n - b_n\} \) and \( \{\frac{a_n}{b_n}\} \)?

8. Using similar ideas what can we say about the convergence of \( \{\ln(a_n)\} \)?