1. State the integral test for series (be as precise as possible and make sure you state the complete theorem in proper English).

**Proof.** Let $f$ be a (1) continuous, (2) positive and (3) decreasing function on $[b, +\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=b}^{\infty} a_n$ is convergent if and only if the improper integral $\int_{b}^{\infty} f(x) \, dx$ is convergent.

2. Now use the integral test to determine convergence or divergence of:

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^3}$$

**Proof.** The function we have to understand is $f(x) = \frac{\ln(x)}{x^3}$. We have to show it satisfies the 3 conditions:

- Continuous: $x^3$ and $\ln(x)$ are continuous and they are both non-zero on $[2, +\infty)$. and so the function is continuous.
- Positivity: $x^3 > 8$ for $x > 2$ and so it is positive. Also $\ln(x)$ is positive for $x > 1$ and so $f(x)$ is positive as the division of positive numbers is positive.
- Decreasing: In order to check if it is decreasing we take the derivative:

$$f'(x) = \frac{1}{2}x^3 - 3x^2 \ln(x) = \frac{x^2 - 3x^2 \ln(x)}{x^6} = \frac{1 - 3 \ln(x)}{x^4}$$

The denominator has only one solution, namely zero, which is not in our interval. The numerator has one root: If $1 - 3 \ln(x) = 0$ then $3 \ln(x) = 1$ then $\ln(x) = \frac{1}{3}$ and so $x = e^{\frac{1}{3}}$. But this number is again smaller than 2 ($e < 8$ and so $e^{\frac{1}{3}} < 8^{\frac{1}{3}} = 2$). So, they are no critical points in the interval. This means that the derivative is always negative and so the function is always decreasing.

So, we can use the integral test for this function, which means we have to determine the convergence of the following integral:

$$\int_{2}^{\infty} \frac{\ln(x)}{x^3} \, dx$$

There are two ways to solve it: Comparison test or substitution. The substitution approach is left to the reader as an exercise (Hint: put $u = \ln(x)$). As for the comparison. Note that
\[
\lim_{x \to \infty} \frac{\ln(x)}{x} = 0. \text{ This means that there exists an } N > 0 \text{ such that for every } x > N \text{ we have } \frac{\ln(x)}{x} < 1 \text{ or, in other words } \ln(x) < x. \text{ Now, if we divide that by } x^3 \text{ we get }
\]
\[
\frac{\ln(x)}{x^3} < \frac{x}{x^3} = \frac{1}{x^2}
\]

So, we split our integral as follows:
\[
\int_2^\infty \frac{\ln(x)}{x^3} \, dx = \int_2^N \frac{\ln(x)}{x^3} \, dx + \int_N^\infty \frac{\ln(x)}{x^3} \, dx
\]

The first one is a proper integral of an integrable function and so it is finite anyway. So, all that matters it the convergence of the second integral. Here we use the mentioned inequality, by the comparison test:
\[
\int_N^\infty \frac{\ln(x)}{x^3} \, dx < \int_N^\infty \frac{1}{x^2} \, dx
\]

But the second integral is convergent by the \( p \)-test \((p = 2 > 1)\). And so the series is convergent as well and we are done.