Midterm 3 Review Sheet:

Time: 6:45-8:00 PM on Monday, April 27th, 2015
Place: Foellinger Auditorium
Sections: 11.1 - 11.11 (all of Chapter 11)

- You have one hour and 15 minutes for the exam but the exam is made for one hour.
- I recommend going over the list of materials mentioned in this sheet, then identifying which topics you are not comfortable with and working on the worksheets related to those topics. You should also solve all the sample exam questions.
- Take your time and go over the worksheets we did in class. In particular, you can work on the questions we didn’t solve in class.
- Half of the exam is multiple choice so make sure you plan your time accordingly.
- Test taking strategy: Go through the multiple choice questions and solve only those you can solve without thinking for long. Then move on to the free response questions and solve them as careful as possible. Then, if you still have extra time go back to the multiple choice questions you skipped.
- Don’t forget to ask me if you have any questions.

Review (Things you should definitely know):

1. **Series and Sequences**: Make sure you know basic notions of sequences: Bounded, increasing, decreasing and convergent. A series is a sum of sequence: \( \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n \), where \( s_n = a_1 + \ldots + a_n \) is the partial sum. There are several tests to determine whether this sum is convergent.

2. **Geometric Series**: The first series is the geometric series, which is of the form:

   \[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \quad \text{if} \ |r| < 1 \]

   (if \( |r| \geq 1 \) then it is divergent).

3. **Divergence Test**: For any sequence \( \{a_n\} \), if \( \lim_{n \to \infty} a_n \neq 0 \) or it is divergent, then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

4. **Telescopic Series**: Let \( \{a_n\} \) be a sequence. Then we get the following telescopic series with limit:

   \[ \sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (a_1 - a_{n+1}) \]

   If \( \{a_n\} \) is convergent (let’s say to \( a \)), then the series is convergent to \( a_1 - a \).
5. **Integral Test**: The integral test only tells us about convergence or divergence and doesn’t give us an actual value. It goes as follows: If \( \{a_n\} \) is a sequence and \( f(x) \) is a function such that \( f(n) = a_n \) and \( f(x) \) is 1) continuous 2) positive 3) decreasing. Then \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if \( \int_1^{\infty} f(x) \, dx \) is convergent.

This gives us the following remainder estimate inequalities:

\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_n^{\infty} f(x) \, dx \quad \text{and} \quad s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_n^{\infty} f(x) \, dx
\]

where \( R_n = a_{n+1} + a_{n+2} + \ldots \).

6. **Comparison Test for Series**: The comparison test is the series version of the integral comparison test. If \( \{a_n\} \) and \( \{b_n\} \) are positive sequences such that \( a_n \leq b_n \). Then: If \( \sum_{n=1}^{\infty} b_n \) is convergent if and only if \( \sum_{n=1}^{\infty} a_n \) is convergent.

7. **Limit Comparison Test**: If \( \{a_n\} \) and \( \{b_n\} \) are positive sequences and if \( \lim_{n \to \infty} \frac{a_n}{b_n} \) is a non-zero finite number, then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are either both convergent or divergent.

8. **Alternating Series**: A series of the form \( \sum_{n=0}^{\infty} (-1)^n b_n \) is called an alternating series. It is convergent if it satisfies the following three conditions:

   I) positive \( (b_n > 0) \)
   II) decreasing \( (b_n > b_{n+1}) \)
   III) converges to zero \( (\lim_{n \to \infty} b_n = 0) \)

If an alternating series satisfies the conditions then we can estimate \( R_n \) by the formula:

\[
|R_n| = \left| \sum_{n=0}^{\infty} (-1)^n b_n - s_n \right| \leq b_{n+1}
\]

So we have two error bound formulas for two different tests. Don’t mix them up.

9. **Root & Ratio Test**: We have two different tests to determine absolute convergence \( \sum_{n=0}^{\infty} |a_n| \) is convergent):

   The ratio test: Let \( L = \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} \). Then if \( L > 1 \) the series is divergent, if \( L < 1 \) the series is absolutely convergent and if \( L = 1 \) then we cannot reach any conclusion.

   The root test: Let \( L = \lim_{n \to \infty} \sqrt[n]{|a_n|} \). Then if \( L > 1 \) our series is divergent, if \( L < 1 \) the series is absolutely convergent and if \( L = 1 \) then we cannot reach any conclusion.
10. **Power Series:** A power series is a function of the form \( f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \). Every power series has a radius of convergence, \( |x-a| < R \), which is equal to:

\[
R = \lim_{n \to \infty} \frac{1}{\sqrt[n]{|a_n|}} \quad \text{(where R can be } \infty \text{)}
\]

11. **Calculus of Power series:** Power series have derivatives and integrals:

\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n(x-c)^n \right) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}
\]

\[
\int \sum_{n=0}^{\infty} a_n(x-c)^n \, dx = \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1} + C
\]

12. **Taylor Series:** Every function \( f(x) \) has associated to it a Taylor series around \( c \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n
\]

and a Taylor series around 0 is called a McLaurin series. There are several McLaurin series you should know (with radius):

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (-1, 1)
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty)
\]

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)
\]

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (-\infty, \infty)
\]

\[
\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \quad (-1, 1)
\]

\[
\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \quad (-1, 1)
\]

\[
(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad \text{(where } \binom{k}{0} = 1 \text{ & } \binom{k}{n} = \frac{k(k-1)...(k-n+1)}{n!}) \quad (-1, 1)
\]

Make sure you memorize them!

13. **Applications of Taylor Series:** Taylor series are used extensively to calculate integrals of some functions. In particular, we use the Taylor polynomial of degree \( n \),

\[
T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n
\]

to approximate the Taylor series and using Taylor’s inequality:

\[
|R_n(x)| = \left| f^{(n+1)}(z) \right| |x-a|^{n+1} \leq \frac{M}{(n+1)!} |x-a|^{n+1}
\]

where \( |f^{(n+1)}(z)| < M \).
Other things to remember:

1. Remember the divergence test is one-sided: The harmonic series is divergent but the sequence $\{\frac{1}{n}\}$ converges to zero.

2. For the integral test you need to remember improper integrals. This means you have to remember how to write them correctly (with limits) and how the comparison test for integrals works, so make sure you go over this again.

3. For the integral test, we need three conditions (continuous, positive and decreasing) which might not be satisfied at the beginning. Then we can just use the integral test on the interval on which it is satisfied. For example, $\frac{x^n}{e^x}$ is decreasing on $[n, +\infty)$ and so have to prove convergence on this interval.

4. The integral test gives us the $p$-test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $p > 1$.

5. For the limit comparison test, make sure $c$ is really non-zero. If $c$ is zero then we cannot reach any conclusion.

6. Remember absolute convergence implies conditional convergence but not vice versa.

7. In order to find McLaurin series try to use derivatives and integrals of familiar McLaurin series rather than trying to find a pattern.

8. The Taylor approximation formula has two independent variables: $z \& x$.

9. Make sure you know all the standard integration formulas. In particular, you should know the integrals of polynomials, exponential functions, trig function and inverse trig functions.

10. Remember to change the bounds of definite integrals when you use substitutions for definite integrals or your answer will be wrong.

11. Make sure you write your final answer in terms of the initial variables in case you used substitution for indefinite integrals.

12. When using the comparison test, remember to actually show that $f(x) \leq g(x)$ before you use the comparison theorem.

13. Don’t forget the $c$ when calculating indefinite integrals.

14. Remember standard facts about algebra as they always pop up, for example:

$$\frac{a}{\frac{b}{c}} = \frac{ad}{bc}$$

15. Remember exponentials and log identities, for example:

$$\log(ab) = \log(a) + \log(b)$$

16. More generally, be careful when using algebra, particularly when dealing with fractions

17. Write careful and complete answers

Good Luck with your third midterm ! I hope you do well !!