Multiple Choice Questions:

1. What number do we get if we approximate the integral \( \int_{0}^{2} x^2 \, dx \) using the trapezoidal method with 4 intervals? Answer: (C)

Proof. The formula of the trapezoidal method is

\[
\frac{b-a}{2n} \left( f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n) \right).
\]

As they are 4 intervals we use the points \( \{0, 0.5, 1, 1.5, 2\} \). We get:

\[
\int_{0}^{2} x^2 \, dx \approx \frac{2-0}{8} (0 + 2 \cdot \frac{1}{4} + 2 \cdot 1 + 2 \cdot \frac{9}{4} + 4) = \frac{11}{4}
\]

\[\square\]

Determine whether each of these series diverge or converge and if it converges to which value:

2. \( \sum_{n=1}^{\infty} \frac{3^{2n+1}}{5^n 2^{2n}} \) Answer: (D)

Proof. After a little bit of algebra we see it turns into a geometric series:

\[
\sum_{n=1}^{\infty} \frac{3^{2n+1}}{5^n 2^{2n}} = 3 \sum_{n=1}^{\infty} \frac{3^{2n}}{5^n 4^n} = 3 \sum_{n=1}^{\infty} \left( \frac{9}{20} \right)^n = 3 \frac{\frac{9}{20}}{1 - \frac{9}{20}} = \frac{27}{11}
\]

\[\square\]

3. \( \sum_{n=5}^{\infty} \frac{6}{9n^2 + 6n - 8} \) Answer: (D)

Proof. After some factorization it turns into a telescopic series:

\[
\sum_{n=5}^{\infty} \frac{6}{9n^2 + 6n - 8} = \sum_{n=5}^{\infty} \frac{6}{(3n-2)(3n+4)}
\]

After using partial fractions we get:

\[
\sum_{n=5}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+4}
\]

We want to make it into a telescopic series, however there is a problem: If we let \( a_n = \frac{1}{3n-2} \) then \( a_{n+1} = \frac{1}{3(n+1)-2} = \frac{1}{3n+1} \) but the second term is not equal to that. So, we add and subtract it to get the following:

\[
\sum_{n=5}^{\infty} \frac{1}{3n-2} - \frac{1}{3n+1} + \frac{1}{3n+1} - \frac{1}{3n+4}
\]

Now, notice that both parts are separately telescopic and so we can solve them independently and add the results:

\[
\sum_{n=5}^{\infty} \left( \frac{1}{3n-2} - \frac{1}{3n+1} \right) + \sum_{n=5}^{\infty} \left( \frac{1}{3n+1} - \frac{1}{3n+4} \right)
\]
Remember the answer of a telescopic series is \( \lim_{n \to \infty} (a_n - a_{n+1}) \) and so the answer is:

\[
\lim_{n \to \infty} \left( \frac{1}{13} - \frac{1}{3n + 1} \right) + \lim_{n \to \infty} \left( \frac{1}{16} - \frac{1}{3n + 4} \right) = \frac{1}{13} + \frac{1}{16} = \frac{29}{208}
\]

Determine whether each of these series diverge or converge:

4. \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n))} \) Answer: Diverges

Proof. We use the integral test to show it diverges. First we show that \( f(x) = \frac{1}{x \ln(x) \ln(\ln(x))} \) satisfies our three conditions. Note for \( x > e \) \( \ln(x) > 1 \) and so \( \ln(\ln(x)) > 0 \) and so the function is positive for \( x > e \). Also, \( x \ln(x) \ln(\ln(x)) = 0 \) if and only if either \( x = 0 \) or \( \ln(x) = 0 \) which means that \( x = 1 \) or \( \ln(\ln(x)) = 0 \) which means that \( x = e \) and so if \( x > e \) then our function has a non-zero denominator. This means \( f(x) \) is continuous on \((e, +\infty)\). Finally, \( x \) an \( \ln(x) \) are both increasing functions. This means that \( \ln(\ln(x)) \) is also increasing (if \( 0 < x_1 < x_2 \) then \( \ln(x_1) < \ln(x_2) \) and then \( \ln(\ln(x_1)) < \ln(\ln(x_2)) \)). This means that \( x \ln(x) \ln(\ln(x)) \) is also increasing (multiplication preserves order). Finally, this means that \( f(x) \) is decreasing. Indeed, if \( 0 < x_1 < x_2 \), then \( x_1 \ln(x_1) \ln(\ln(x_1)) < x_2 \ln(x_2) \ln(\ln(x_2)) \) so if we reverse it we get:

\[
\frac{1}{x_2 \ln(x_2) \ln(\ln(x_2))} < \frac{1}{x_1 \ln(x_1) \ln(\ln(x_1))}
\]

and so it is decreasing. So, \( f(x) \) satisfies all three conditions on the interval \((e, +\infty)\) (we take \( e \) so that the left boundary is not improper and later calculations are easier). So, our series is divergent if and only if \( \int_{e^ e}^{\infty} \frac{1}{x \ln(x) \ln(\ln(x))} \, dx \) is divergent. We now calculate the improper integral:

\[
\int_{e^ e}^{\infty} \frac{1}{x \ln(x) \ln(\ln(x))} \, dx = \lim_{t \to \infty} \int_{e^ e}^{t} \frac{1}{x \ln(x) \ln(\ln(x))} \, dx
\]

We substitute \( u = \ln(x) \) then \( du = \frac{dx}{x} \) and the boundaries are \( \ln(e^ e) = e \) and \( \ln(t) \) which we call \( s \). As \( t \) converges to \( \infty \), \( s \) also converges to \( \infty \). Then:

\[
\lim_{t \to \infty} \int_{e^ e}^{t} \frac{1}{x \ln(x) \ln(\ln(x))} \, dx = \lim_{s \to \infty} \int_{e}^{s} \frac{1}{u \ln(u)} \, du
\]

Now, we do the same substitution again. Let \( w = \ln(u) \) then \( dw = \frac{du}{u} \) and the boundaries are \( \ln(e) = 1 \) and \( \ln(s) \) which we call \( r \). As \( s \) converges to \( \infty \), \( r \) also converges to \( \infty \). Then:

\[
\lim_{s \to \infty} \int_{e}^{s} \frac{1}{u \ln(u)} \, du = \lim_{r \to \infty} \int_{1}^{r} \frac{1}{w} \, dw = \lim_{r \to \infty} \ln(w)|_1^r \lim_{r \to \infty} \ln(r)
\]

And this sequence goes to \( \infty \) and so the integral is divergent and so, by the integral test, the series is divergent.

5. \( \sum_{n=2}^{\infty} \frac{7n}{n^3 - 4n^2 + 2} \) Answer: Converges
**Proof.** It converges by the limit comparison test. Note that for \( n > 4, n^3 > 4n^2 \) and so \( n^3 - 4n^2 > 0 \) and so \( n^3 - 4n^2 + 2 > 0 \). So, we will show \( \sum_{n=4}^{\infty} \frac{7n}{n^3 - 4n^2 + 2} \) converges and this implies that the initial series converges. As we stated above the sequence \( a_n = \frac{7n}{n^3 - 4n^2 + 2} \) is positive. Let \( b_n = \frac{1}{n^2} \) which is also positive. Note \( b_n \) converges by the \( p \)-test as \( p = 2 > 1 \). Now, we calculate \( \lim_{n \to \infty} \frac{a_n}{b_n} \):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{7n}{n^3 - 4n^2 + 2}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{7n^3}{n^3 - 4n^2 + 2} = 7
\]

As the answer is positive and not infinite, by the limit comparison test \( \sum_{n=2}^{\infty} \frac{7n}{n^3 - 4n^2 + 2} \) converges.

6. \( \sum_{n=0}^{\infty} \frac{1}{n^2 - 1} \) Answer: Diverges

**Proof.** This series is not defined for \( n = 1 \) and so the series diverges.

7. \( \lim_{n \to \infty} \frac{n!}{4^n} \) Answer: Diverges

**Proof.** The point is that for large enough \( n, 5^n < n! \). As a matter of fact \( 5^{125} < 125! \) as we have:

\[
5^{125} = 5^{120} \cdot 25 \cdot 125 < 5 \cdot 6 \cdot 7 \cdot 24 \cdot 26 \cdot \ldots \cdot 124 \cdot 25 \cdot 125 < 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \cdot 24 \cdot 25 \cdot 26 \cdot \ldots \cdot 124 \cdot 125 = 125!
\]

and for \( n > 5 \) if \( 5^n < n! \) then \( 5^{n+1} = 5^n \cdot 5 < n! \cdot 5 < n! \cdot (n + 1) = (n + 1)! \) And so for \( n > 125 \) we have:

\[
\frac{n!}{4^n} > \frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n
\]

However, this is just an exponential function with the base bigger than one and so it goes to infinity, which means it is divergent. By comparison then \( \frac{n!}{4^n} \) is also divergent.

8. \( \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \) Answer: Diverges

**Proof.** Remember that:

\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

This means that we have the following:

\[
\lim_{y \to \infty} \frac{\sin\left(\frac{1}{y}\right)}{\frac{1}{y}} = 1
\]

This follows from the previous fact if we substitute \( x = \frac{1}{y} \) and realize that if \( y \) goes to \( \infty \) then \( x \) goes to zero.

With this we can solve the question using limit comparison. Notice that \( 0 < \frac{1}{n} < 1 \) and so \( a_n = \sin\left(\frac{1}{n}\right) > 0 \). Let \( b_n = \frac{1}{n} \) which is also positive. This series is divergent as it is the harmonic series. Now, we use the limit comparison test:

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1
\]
This limit (by the statement above) converges to 1, which is finite non-zero positive number and so the series diverges.

9. True or False: If \( \sum_{n=1}^{\infty} a_n \) converges, then the sequence \( \{s_n\}_{n=1}^{\infty} \) converges, \( (s_n = a_1 + a_2 + \ldots + a_n) \).

Answer: True

Proof. This is true by definition as we know \( \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n \).

Free Response:

10. Consider the lamina \( L \) in the plane of constant density \( \rho \), which is bounded by the curves 
\[ x = 5 - y^4, \quad x = y^2 - 1. \]

Find the moments \( M_x \) and \( M_y \) and the center of mass of \( L \).

Proof. Note that both functions are even and so the shape is symmetric with respect to the \( x \)-axis. This implies that \( \bar{y} = 0 \) which also means that \( M_x = 0 \). So, all we have to do is to find \( M_y \) and then \( \bar{x} \). We first have to find the bounds of \( y \) by solving \( 5 - y^4 = y^2 - 1 \).

\[ 5 - y^4 = y^2 - 1 \implies y^4 + y^2 - 6 = 0 \implies (y^2 + 3)(y^2 - 2) = 0 \]

But \( y^2 + 3 \) is never zero and so \( y^2 - 2 = 0 \) and so \( y \) is between \( \sqrt{2} \) and \( -\sqrt{2} \).

Now, we calculate:

\[ M_x = \rho \int_{a}^{b} \frac{1}{2} ([f(y)]^2 - [g(y)]^2) \, dy = \rho \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{2} ([5 - y^4]^2 - [y^2 - 1]^2) \, dy = \]

\[ \rho \int_{0}^{\sqrt{2}} 25 - 10y^4 + y^8 - y^4 + 2y^2 - 1 \, dy = \rho \int_{0}^{\sqrt{2}} y^8 - 11y^4 + 2y^2 + 24 \, dy = \]

\[ \rho \left( \frac{1}{9} y^9 - \frac{11}{5} y^5 + \frac{2}{3} y^3 + 24y \right) \bigg|_{0}^{\sqrt{2}} = \rho \left( \frac{1}{9} (\sqrt{2})^9 - \frac{11}{5} (\sqrt{2})^5 + \frac{2}{3} (\sqrt{2})^3 + 24(\sqrt{2}) \right) = \]

\[ \sqrt{2} \rho \left( \frac{1}{9} (\sqrt{2})^8 - \frac{11}{5} (\sqrt{2})^4 + \frac{2}{3} (\sqrt{2})^2 + 24 \right) = \sqrt{2} \rho \left( \frac{16}{9} - \frac{44}{5} + \frac{4}{3} + 24 \right) = \frac{824\sqrt{2}\rho}{45} \]

Now, all we need is the area to find \( \bar{x} \). The area is:

\[ A = \int_{-\sqrt{2}}^{\sqrt{2}} (5 - y^4) - (y^2 - 1) \, dy = 2 \int_{0}^{\sqrt{2}} 6 - y^4 - y^2 \, dy = \]

\[ 12y - \frac{2}{5} y^5 - \frac{2}{3} y^3 \bigg|_{0}^{\sqrt{2}} = 12(\sqrt{2}) - \frac{2}{5} (\sqrt{2})^5 - \frac{2}{3} (\sqrt{2})^3 = \sqrt{2}(12 - \frac{8}{5} - \frac{4}{3}) = \frac{136\sqrt{2}}{15} \]

And so

\[ \bar{x} = \frac{M_x}{\rho A} = \frac{824\sqrt{2}\rho}{45} \cdot \frac{45}{136\sqrt{2}} = \frac{103}{51} \]

As we said \( \bar{y} = 0 \) and so

\( (\bar{x}, \bar{y}) = (\frac{103}{51}, 0) \)
11. We have a triangle shaped swimming pool formed by the lines \( y = -2x \) \( y = 0 \) and \( x = 5 \), which is completely filled up with water. Draw a picture of the pool and find the hydraulic force of the water at the bottom.

Proof. If you draw the shape correctly you get a right triangle with base of length 5 on the \( x \)-axis height equal to \(-10\). So the length of each strip is \( 5 - \frac{y}{2} = 5 + \frac{y}{5} \) where \(-10 \leq y \leq 0\). So, we get the following integral we have to calculate:

\[
F = \int_{-10}^{0} 1000 \cdot 9.8 \cdot (0 - y)(5 + \frac{y}{5}) \, dy = -9800 \int_{-10}^{0} 5y + \frac{y^2}{5} \, dy = -9800\left(\frac{5}{2}y^2 + \frac{1}{15}y^3\right)|_{-10}^{0} = 0 + 9800\left(\frac{5}{2}(-10)^2 + \frac{1}{15}(-10)^3\right) = \frac{5390000}{3}
\]

12. Let \( y = \sqrt{x - 1} \) from \( x = 1 \) to \( x = 10 \). Set up but do not evaluate the following:

a) The arc length of the curve
b) The surface area when rotated around the \( x \)-axis where the integral has to be in terms of \( x \).
c) The surface area when rotated around the \( x \)-axis where the integral has to be in terms of \( y \).
d) The surface area when rotated around the \( y \)-axis.

Proof. We just plug into the formulas for arc length and surface area. The only other information we need is the function in terms of \( x \) so that we can later write the answer in terms of \( y \). So, we find the function which is: \( x = y^2 + 1 \) from \( y = 0 \) to \( y = 3 \).

\[
a) \int_{a}^{b} ds = \int_{1}^{10} \sqrt{1 + \left(\frac{1}{2\sqrt{x - 1}}\right)^2} \, dx = \int_{1}^{10} \sqrt{1 + \frac{1}{4x - 4}} \, dx \\
b) 2\pi \int_{a}^{b} yds = 2\pi \int_{1}^{10} \sqrt{x - 1} \sqrt{1 + \left(\frac{1}{2\sqrt{x - 1}}\right)^2} \, dx = 2\pi \int_{1}^{10} \sqrt{x - 1} \sqrt{1 + \frac{1}{4x - 4}} \, dx \\
c) 2\pi \int_{a}^{b} yds = 2\pi \int_{0}^{3} y\sqrt{(2y)^2 + 1} \, dy = 2\pi \int_{0}^{3} y\sqrt{4y^2 + 1} \, dy \\
d) 2\pi \int_{a}^{b} xds = 2\pi \int_{1}^{10} x \sqrt{1 + \left(\frac{1}{2\sqrt{x - 1}}\right)^2} \, dx = 2\pi \int_{1}^{10} x \sqrt{1 + \frac{1}{4x - 4}} \, dx
\]

13. Suppose the sum of the series \( s = \sum_{k=1}^{\infty} \frac{1}{k^3} \) is approximated by its 5th partial sum, \( s_5 = 1 + \frac{1}{8} + \ldots + \frac{1}{125} \). Approximate the maximum possible error in this estimation.

Proof. Our goal is to find an upper bound for \( R_5 \) we know that \( R_5 < \int_{5}^{\infty} \frac{1}{x^3} \), because the function \( \frac{1}{x^3} \) satisfies the conditions of the integral test (it is positive, decreasing and continuous). So, we just find that integral:

\[
\int_{5}^{\infty} \frac{1}{x^3} \, dx = \lim_{t \to \infty} \int_{5}^{t} \frac{1}{x^3} \, dx = \lim_{t \to \infty} \frac{1}{-2x^2} \bigg|_{5}^{t} = \lim_{t \to \infty} \frac{1}{-2t^2} - \frac{1}{-2 \cdot 25} = \frac{1}{50}
\]