Final Exam Review Sheet:

Time: 1:30-4:30 PM on Monday, May 11th, 2015
Place: Foellinger Auditorium
Sections: 7.1-7.5, 7.7-7.8, 8.1 - 8.3, 10.1-10.4, 11.1 - 11.11

• You have three hours for the exam.

• I recommend going over the list of materials mentioned in this sheet, then identifying which topics you are not comfortable with and working on the worksheets related to those topics. You should also solve all the sample exam questions.

• Take your time and go over the worksheets we did in class. In particular, you can work on the questions we didn’t solve in class.

• Half of the exam is multiple choice so make sure you plan your time accordingly.

• Test taking strategy: Go through the multiple choice questions and solve only those you can solve without thinking for long. Then move on to the free response questions and solve them as careful as possible. Then, if you still have extra time go back to the multiple choice questions you skipped.

• Don’t forget to ask me if you have any questions.

Review (Things you should definitely know):

The topics are split into three parts.

Part I (Midterm 1):

1. **Substitution**: Although not officially part of the syllabus as it is in chapter 6 yet I think it is the cornerstone of everything we did in part I. Make sure you are comfortable with substitution techniques. This is very important. If you are not then you should most certainly work on it. You can also come and ask me if you think you need some help.

2. **Integration by Parts**: Remember the formula for the integration by parts:

\[ \int udv = uv - \int vdu \]

You can use it very often when we have a product of a polynomial and a trig (or exponential) function. You need to practice a lot so that you can identify \( u \) and \( v \) correctly, however, if you are not sure you can always try to use this formula:

**LIATE**

which stands for logarithmic (L), inverse trigonometric (I), algebraic (A), trigonometric (T), and exponential (E) and the rule is:

*Pick \( u \) in the order of LIATE.*

However, remember there are always exceptions.
3. **Trig Integrals:** There are several standard trig integrals:

\[ \int \sin^n x \cos^m x \, dx \]

If either \( n \) or \( m \) are odd, then you have to use substitution.

If \( n \) or \( m \) are both even, then we have to use the following formulas:

\[
\begin{align*}
\sin^2 x &= \frac{1}{2} (1 - \cos 2x) \\
\cos^2 x &= \frac{1}{2} (1 + \cos 2x) \\
\sin x \cos x &= \frac{1}{2} \sin 2x
\end{align*}
\]

\[ \int \tan^m x \sec^n x \, dx \]

If \( m \) is odd or \( n \) is even, then use substitution.

If \( m \) is even and \( n \) is odd then there is no easy way. (But you should remember \( \int \sec x \, dx \) and \( \int \sec^3 x \, dx \)).

There are trig integrals which are not of this form where determining the correct substitution needs some experience, see question 3 of worksheet 3.

4. **Trig Substitution:** Some integrals can be solved by substituting a trig function. These are the following cases:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Substitution</th>
<th>( dx )</th>
<th>Identity</th>
<th>Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin \theta )</td>
<td>( dx = a \cos \theta , d\theta )</td>
<td>( a^2 - a^2 \sin^2 \theta = a^2 \cos^2 \theta )</td>
<td>(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})</td>
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<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan \theta )</td>
<td>( dx = a \sec^2 \theta , d\theta )</td>
<td>( a^2 + a^2 \tan^2 \theta = a^2 \sec^2 \theta )</td>
<td>(-\frac{\pi}{2} \leq \theta &lt; \frac{\pi}{2})</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec \theta )</td>
<td>( dx = a \sec \theta \tan \theta , d\theta )</td>
<td>( a^2 \sec^2 \theta - a^2 = a^2 \tan^2 \theta )</td>
<td>( 0 \leq \theta &lt; \frac{\pi}{2})</td>
</tr>
</tbody>
</table>

So, whenever you encounter these cases you can use trig substitution. However, you have to be careful as there are exceptions. Also, sometimes you have to first complete a square before you can use trig sub.

5. **Partial Fractions:** We introduce techniques to solve integrals of the form \( \frac{P}{Q} \) such that \( P \) and \( Q \) are polynomials. First, we have to use polynomial division to make sure the degree of \( P \) is lower than \( Q \). Then we split \( Q = Q_1 Q_2 \ldots Q_n \) into its irreducible factors which can have three forms 1) \( x - a \) or 2) \( (x - a)^n \) (where \( n \) positive) or 3) \( ax^2 + bx + c \) (where \( b^2 - 4ac < 0 \)). Depending on the form we now using following partial fraction equation:

- If \( Q_i = x - a \) then we add \( \frac{A}{x-a} \)
- If \( Q_i = (x - a)^n \) then we add \( \frac{A_n}{(x-a)^n} + \ldots + \frac{A_1}{x-a} \)
- If \( Q_i = ax^2 + bx + c \) where \( b^2 - 4ac < 0 \) then we add \( \frac{C x + D}{ax^2 + bx + c} \)

For example:

\[
\frac{P(x)}{(x-a)(x-b)^2(x^2 + 1)} = \frac{A}{x - a} + \frac{B_2}{(x-b)^2} + \frac{B_1}{x - b} + \frac{C x + D}{x^2 + 1}
\]

From there on we use \( \ln \) and \( \arctan \) to solve the integrals.
6. **Improper Integrals:** There are two general types:

**Type 1:**
- \( \int_{a}^{\infty} f(x) \, dx = \lim_{t \to \infty} \int_{a}^{t} f(x) \, dx \)
- \( \int_{-\infty}^{b} f(x) \, dx = \lim_{t \to -\infty} \int_{t}^{b} f(x) \, dx \)
- \( \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{c} f(x) + \int_{c}^{\infty} f(x) \, dx \)

**Type 2:**
- \( \int_{a}^{b} f(x) \, dx = \lim_{t \to b^{-}} \int_{a}^{t} f(x) \, dx \) if \( f \) discontinuous at \( b \)
- \( \int_{a}^{b} f(x) \, dx = \lim_{t \to a^{+}} \int_{t}^{b} f(x) \, dx \) if \( f \) discontinuous at \( a \)
- \( \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) + \int_{c}^{b} f(x) \, dx \) if \( f \) discontinuous at \( c \)

There are some improper integrals which can’t be solved so easily. For those cases we have some comparison theorems: Let \( f \) and \( g \) be continuous functions. Suppose that for all \( x \geq a, 0 \leq f(x) \leq g(x) \). Then:

- If \( \int_{a}^{\infty} g(x) \, dx \) is convergent, then so is \( \int_{a}^{\infty} f(x) \, dx \), and \( \int_{a}^{\infty} f(x) \, dx \leq \int_{a}^{\infty} g(x) \, dx \);
- if \( \int_{a}^{\infty} f(x) \, dx \) is divergent, then so is \( \int_{a}^{\infty} g(x) \, dx \).

7. **Strategies:** This list is not exhaustive and just covers general ideas. It is very important to practice a lot and gain as much experience as possible. There are many integrals which don’t exactly fall into any of the mentioned categories. Also, many integrals require the use of several methods.

**Part II (Midterm 2):**

8. **Approximating Integrals:** There are some integrals we cannot calculate easily and so we can use approximation formulas.

The Trapezoidal Rule:

\[
\int_{a}^{b} f(x) \, dx = \frac{b-a}{2n} (f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))
\]

Simpson’s Rule:

\[
\int_{a}^{b} f(x) \, dx = \frac{b-a}{3n} (f(x_0) + 4f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + 4f(x_{n-1}) + f(x_n))
\]

where \( x_i = a + i\Delta x \) and in the second case \( n \) has to be even.
9. **Arc Length and Surface Area**: The general formula for arc length is \( \int_a^b ds \), where \( ds = \sqrt{dx^2 + dy^2} \). Depending on the function \( ds \) has to be interpreted in the correct way. If \( y = f(x) \) then \( dx = 1 \) and \( dy = f'(x) \) and so \( ds = \sqrt{1 + (f'(x))^2} dx \). If \( x = g(y) \) then \( dy = 1 \) and \( dx = g'(y) \) and so \( ds = \sqrt{1 + (g'(y))^2} dy \).

The surface area is calculated in a very similar fashion. We only calculate it for surfaces which come from rotations. The general formula is \( 2\pi x ds \) (if rotated around the y-axis) or \( 2\pi y ds \) (if rotated around the x-axis).

10. **Hydrostatic Force**: The formula for hydrostatic force is:

\[
F = mg = \rho g A d
\]

where \( F \) is force, \( m \) mass, \( g \) standard gravity acceleration (9.8 m/s\(^2\) = 32 ft/s\(^2\)), \( \rho \) density of the water (1000 kg/m\(^3\)), \( A \) area and \( d \) distance. As we know \( P = \frac{F}{A} \) where \( P \) is pressure, which gives us:

\[
P = \rho g d
\]

As \( \rho \) and \( g \) are constant all we have to do is to find a formula for the distance and area of the shape and take integral.

11. **Moments and Centroids**: Every shape determined by two functions \( f(x) \) and \( g(x) \) has two moments:

\[
M_x = \rho \int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2) \, dx \quad \text{and} \quad M_y = \rho \int_a^b x(f(x) - g(x)) \, dx
\]

where \( M_x \) is the moment of the system around the \( x \)-axis and \( M_y \) around the \( y \)-axis. Based on this we can find the centroid:

\[
\bar{x} = \frac{M_y}{m} = \frac{1}{A} \int_a^b x(f(x) - g(x)) \, dx \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{1}{A} \int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2) \, dx
\]

where the centroid is \((\bar{x}, \bar{y})\).

12. **Series and Sequences**: Make sure you know basic notions of sequences: Bounded, increasing, decreasing and convergent. A series is a sum of sequence: \( \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n \), where \( s_n = a_1 + ... + a_n \) is the partial sum. There are several tests to determine whether this sum is convergent.

13. **Geometric Series**: The first series is the geometric series, which is of the form:

\[
\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \quad \text{if} \quad |r| < 1
\]

(if \( |r| \geq 1 \) then it is divergent).

14. **Divergence Test**: For any sequence \( \{a_n\} \), if \( \lim_{n \to \infty} a_n \neq 0 \) or it is divergent, then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.
15. **Telescopic Series**: Let \( \{a_n\} \) be a sequence. Then we get the following telescopic series with limit:

\[
\sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (a_1 - a_{n+1})
\]

If \( \{a_n\} \) is convergent (let’s say to \( a \)), then the series is convergent to \( a_1 - a \).

16. **Integral Test**: The integral test only tells us about convergence or divergence and doesn’t give us an actual value. It goes as follows: If \( \{a_n\} \) is a sequence and \( f(x) \) is a function such that \( f(n) = a_n \) and \( f(x) \) is

I) continuous
II) positive
III) decreasing

Then \( \sum_{n=1}^{\infty} a_n \) is convergent if and only if \( \int_{1}^{\infty} f(x) \, dx \) is convergent.

This gives us the following remainder estimate inequalities:

\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx \quad \text{and} \quad s_n + \int_{n+1}^{\infty} f(x) \, dx \leq s \leq s_n + \int_{n}^{\infty} f(x) \, dx
\]

where \( R_n = a_{n+1} + a_{n+2} + \ldots \).

17. **Comparison Test for Series**: The comparison test is the series version of the integral comparison test:

If \( \{a_n\} \) and \( \{b_n\} \) are positive sequences such that \( a_n \leq b_n \). Then:

I) If \( \sum_{n=1}^{\infty} b_n \) is convergent then \( \sum_{n=1}^{\infty} a_n \) is also convergent.

II) If \( \sum_{n=1}^{\infty} a_n \) is divergent then \( \sum_{n=1}^{\infty} b_n \) is also divergent.

18. **Limit Comparison Test**: If \( \{a_n\} \) and \( \{b_n\} \) are positive sequences and if

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = c
\]

where \( c \) is a non-zero number. Then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are either both convergent or divergent.

**Part III (Midterm 3):**

19. **Alternating Series**: A series of the form \( \sum_{n=0}^{\infty} (-1)^n b_n \) is called an alternating series.

It is convergent if it satisfies the following three conditions:
I) positive \((b_n > 0)\)
II) decreasing \((b_n > b_{n+1})\)
III) converges to zero \((\lim_{n \to \infty} b_n = 0)\)

If an alternating series satisfies the conditions then we can estimate \(R_n\) by the formula:

\[
|R_n| = |\sum_{n=0}^{\infty} (-1)^n b_n - s_n| \leq b_{n+1}
\]

So we have two error bound formulas for two different tests. Don’t mix them up.

20. **Root & Ratio Test:** We have two different tests to determine absolute convergence \((\sum_{n=0}^{\infty} |a_n| \text{ is convergent})\):

*The ratio test:* Let \(L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}\). Then:

I) If \(L > 1\), the series is divergent
II) If \(L < 1\), the series is absolutely convergent
III) If \(L = 1\), we cannot reach any conclusion.

*The root test:* Let \(L = \lim_{n \to \infty} \sqrt[n]{|a_n|}\). Then:

I) If \(L > 1\), the series is divergent
II) If \(L < 1\), the series is absolutely convergent
III) If \(L = 1\), we cannot reach any conclusion.

21. **Power Series:** A power series is a function of the form

\[
f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n.
\]

Every power series has a radius of convergence, \(|x - a| < R\), which is equal to:

\[
R = \lim_{n \to \infty} \frac{1}{\sqrt[n]{|a_n|}} \text{(where R can be } \infty)\]

22. **Calculus of Power Series:** Power series have derivatives and integrals:

\[
\frac{d}{dx} \left( \sum_{n=0}^{\infty} a_n (x - c)^n \right) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1}
\]

\[
\int \left( \sum_{n=0}^{\infty} a_n (x - c)^n \right) dx = \sum_{n=0}^{\infty} \frac{a_n (x - c)^{n+1}}{n+1} + C
\]
23. **Taylor Series**: Every function \( f(x) \) has associated to it a Taylor series around any point \( c \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n
\]

A Taylor series around 0 is called a McLaurin series. There are several McLaurin series you should know (with radius):

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (-1, 1)
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (-\infty, \infty)
\]

\[
\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (-\infty, \infty)
\]

\[
\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (-\infty, \infty)
\]

\[
\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad (-1, 1)
\]

\[
\ln(x + 1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{(n)} \quad (-1, 1)
\]

\[
(1 + x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad (\text{where } \binom{k}{0} = 1 & \binom{k}{n} = \frac{k(k-1)...(k-n+1)}{n!} \quad (-1, 1)
\]

Make sure you memorize them!

24. **Applications of Taylor Series**: Taylor series are used extensively to calculate integrals of some functions. Sometimes it is better to approximate our series with the Taylor polynomial of degree \( n \):

\[
T_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + ... + \frac{f^{(n)}(c)}{n!}(x - c)^n
\]

the difference between the series and the polynomial (the error term) is given by Taylor’s inequality:

\[
|R_n(x)| = \frac{|f^{(n+1)}(z)|}{(n+1)!}|x - a|^{n+1} \leq \frac{M}{(n+1)!}|x - a|^{n+1}
\]

where \(|f^{(n+1)}(z)| < M\).

Part IV (Final Part):

25. **Parametric Equations**: We use parametric equations to deal with complicated functions. A parametrization is of the form \( x = f(t) \) and \( y = g(t) \). Using the chain rule we can use the parametrization to find derivatives:

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}
\]
We can use it to find the area under the curve:
\[
\int y \, dx = \int g(t) f'(t) \, dt
\]

We can also use it to find arc length (if \( a \leq t \leq b \)):
\[
\int_a^b ds = \int_a^b \sqrt{dx^2 + y^2} = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} \, dt
\]

We can also use it to find surface area of the rotation around the \( x \)-axis:
\[
\int_a^b 2\pi y ds = \int_a^b 2\pi g(t) \sqrt{(f'(t))^2 + (g'(t))^2} \, dt
\]

and also surface area around the \( y \)-axis:
\[
\int_a^b 2\pi x ds = \int_a^b 2\pi f(t) \sqrt{(f'(t))^2 + (g'(t))^2} \, dt
\]

26. **Polar Coordinates**: Polar coordinates are a new way to represent points. It makes certain complicated equation a lot simpler and helps us to do calculus with it. For example the polar coordinates of the unit circle is \( r = 1 \). We use the following conversions to change between polar and cartesian coordinates:

\[
x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta)
\]

Opposite conversions:

\[
r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)
\]

Using those we have the derivative:

\[
\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx} = \frac{dr}{d\theta} \sin(\theta) + r \cos(\theta) - \frac{dr}{d\theta} \cos(\theta) - r \sin(\theta)
\]

We have the following formula for the area under the curve (for \( \alpha \leq \theta \leq \beta \)):
\[
\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta
\]

Also, we have the following arc length formula:
\[
\int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta
\]
Other things to remember:

1. Make sure you know all the standard integration formulas. In particular, you should know the integrals of polynomials, exponential functions, trig function and inverse trig functions.

2. Remember to change the bounds of definite integrals when you use substitutions for definite integrals or your answer will be wrong.

3. Make sure you write your final answer in terms of the initial variables in case you used substitution for indefinite integrals.

4. Remember there are exceptions to LIATE like question 4 on worksheet 2.

5. Make sure you know how to complete a square.

6. For trig substitutions the expressions don’t have to be under a square root. There are cases where trig sub helps for $x^2 + a^2$ (as you can see on question 3 (c) of worksheet 4).

7. There are some occasions where it seems trig sub should work but it actually makes it harder: look at question 7 of worksheet 4.

8. When writing your partial fractions then make sure you use different variables for every constant.

9. Be careful with your algebra when solving partial fractions. Remember if the numerator is of the form $Ax + B$ then you have to split the integral to make one part into ln and one into arctan.

10. Remember how to use l’Hospitals rule and other limit techniques as you need them for improper integrals.

11. Remember the difference between integral and area.

12. Don’t forget the $c$ when calculating indefinite integrals.

13. Make sure you remember the shell and disc method from Calc 1.

14. Make sure you memorize the two different formulas for approximating integrals as they will not be provided.

15. Make sure you identify the bounds correctly when you want to calculate the arc length.

16. Make sure you have replaced all variables by one single variable before you actually take integrals, when dealing with arc length, surface area or hydrostatic force.

17. When finding hydrostatic forces use symmetries to simplify your calculations.

18. Remember the divergence test is one-sided: The harmonic series is divergent but the sequence $\{\frac{1}{n}\}$ converges to zero.

19. There are many sequences which can be solved using the telescopic series but first need some algebraic manipulations to get the right shape. For example, $\ln\left(\frac{n}{n+1}\right)$ or $\frac{1}{4n^2-1}$ (the second one needs partial fractions).
20. When using the comparison test, remember to actually show that \( f(x) \leq g(x) \) before you use the comparison theorem.

21. For the limit comparison test, make sure \( c \) is really non-zero. If \( c \) is zero then we cannot reach any conclusion.

22. For the integral test, we need three conditions (continuous, positive and decreasing) which might not be satisfied at the beginning. Then we can just use the integral test on the interval on which it is satisfied. For example, \( \frac{x^n}{e^x} \) is decreasing on \([n, +\infty)\) and so have to prove convergence on this interval.

23. The integral test gives us the \( p \)-test: \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if and only if \( p > 1 \).

24. Remember absolute convergence implies convergence but not vice versa.

25. Remember the root and ratio test only tell you about absolute convergence.

26. In order to find McLaurin series try to use derivatives and integrals of familiar McLaurin series rather than trying to find a pattern.

27. The Taylor approximation formula has two independent variables: \( z \& x \), which means you have to maximize for those two variables independently.

28. A good parametrization can very much simplify calculations. For example the ellipse \( \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = 1 \) has the following good parametrization \( x = a \cos(t) \) and \( y = b \sin(t) \).

29. A polar coordinate with negative radius, corresponds to the opposite side of that angle. For example the point \( (3, \frac{\pi}{2}, -5) \) is the same as the point \((\frac{\pi}{2}, 5)\).

30. Remember standard facts about algebra as they always pop up, for example:

\[
\frac{a}{\frac{b}{d}} = \frac{ad}{bc}
\]

31. Remember exponentials and log identities, for example:

\[
\log(ab) = \log(a) + \log(b)
\]

32. More generally, be careful when using algebra, particularly when dealing with fractions.

33. Write careful and complete answers

Good Luck with your final exam! I hope you do well!!!