Study Goals: • Use power series to represent functions
• Use power series representation to compute integrals

1. You are given that \( \sum_{n=0}^{\infty} c_n (2)^n \) converges, and that \( \sum_{n=0}^{\infty} c_n 7^n \) diverges.

a) What are the possible values of the radius of convergence of the power series \( \sum c_n x^n \)? Use the following numberline to specify on which intervals it has to converge, has to diverge and which ones we don’t know:

What can you say about the convergence/divergence of the following series?

b) \( \sum c_n (7)^n \)
c) \( \sum c_n (-8)^n \)
d) \( \sum c_n (-2)^n \)

2. In this problem we find an approximation for \( \int_0^1 x \arctan x \, dx \).

a) For each function, write down the first four terms of the power series. State the radius of convergence as \( |x| < R \).

- \( \frac{1}{1-x} \)
- \( \frac{1}{1+x^2} \)
- \( \arctan x \)
- \( x \arctan x \)

b) Write down a series for \( \int_0^1 x \arctan x \, dx \) (for which variable you plug in the bounds 0 and 1? x or n?).
c) Write down enough terms to approximate $\int_0^1 x \arctan x \, dx$ to within $\frac{1}{100}$, and show that you are correct. (Hint: your series should be alternating.)

3. Given $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, compute the power series for
   - $e^{x^2}$
   - $\int e^{x^2} \, dx$

   Is the series for $\int_0^1 e^{x^2} \, dx$ an alternating series?

4. Now go through the same steps to find $\int_0^1 \cos(x^3) \, dx$ using the fact that $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
5. Augustin-Jean Fresnel (1788-1827) was an engineer, mathematician and the French commissioner of lighthouses. He is famous for his work in optics and for developing the Fresnel lens. Originally developed for lighthouses, Fresnel lenses are still used today in many consumer items including computer and overhead projectors. The integral
\[ \int_0^1 \frac{\sin(x)}{x} \, dx \]
occurs in Fresnel’s theory of diffraction, and is known as a Fresnel integral.

(a) Use the power series \( \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \) to evaluate the Fresnel integral as an infinite series.

(b) Estimate the Fresnel integral to within \( 10^{-3} \). (Remember error test for alternating series)

6. Find a power series representation for each of the following functions. Use summation notation, and give the radius of convergence as \( |x| < R \).

a) \( \frac{x}{(1 + 2x)^2} \)

b) \( \ln(1 + 3x^2) \)
7. In the next few problems we approximate $\ln(x)$.

a) Recall linear approximation: $f(x) \approx f(a) + f'(a)(x-a)$. Use linear approximation to estimate $\ln(1.1)$, using $a = 1$.

b) Compute the power series for $\ln(1 + x)$. (Remember $\left(\frac{d}{dx}\right)\ln(x + 1) = \frac{1}{x + 1}$)

c) Use the series to approximate $\ln(1.1)$ to within 1/1000.

d) Compute the power series for $\ln(y)$ by making the change of variable $y = x + 1$. Where is the series centered? What is the interval of convergence?