One last time: Determine if the following series converge absolutely, converge conditionally, or diverge.

1. \[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \ln n}{\sqrt{n}}. \]

2. \[ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}. \]

3. \[ \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}. \]

**Power Series:** A power series is a function of the form

\[ f(x) = \sum_{n=1}^{\infty} a_n x^n \]

Woah, what’s going on? We now have two variables: \( x \) and \( n \). Let’s compare them.

4. For each sentence circle whether it relates to \( x \) or \( n \):
   a) It is a continuous variable.  
   b) It is a discrete variable.  
   c) It is restricted when we want to approximate the series.  
   d) We use it when we want to take derivative or integral.
Every power series has a radius of convergence $r$ which determines where the function is defined (and so even makes sense in the first place). So we really care about finding that radius of convergence.

5. You are given a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ and you know that $\sum_{n=0}^{\infty} c_n (-3)^n$ converges, and that $\sum_{n=0}^{\infty} c_n 5^n$ diverges. Use the following numberline to specify on which intervals $f(x)$ has to converge, has to diverge and which ones we don’t know:

![Numberline](image)

What can you say about the convergence/divergence of the following series?

a) $\sum_{n=0}^{\infty} c_n (-6)^n$

b) $\sum_{n=0}^{\infty} c_n 2^n$

c) $\sum_{n=0}^{\infty} c_n 4^n$

d) $\sum_{n=0}^{\infty} c_n (-5)^n$

Use the ratio test to determine the radius of convergence. Then determine the interval of convergence.

6. $\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$.

7. $\sum_{n=1}^{\infty} \frac{(-1)^n (x - 3)^n}{n \cdot 5^n}$. 
8. \[ \sum_{n=1}^{\infty} (-1)^n \frac{(x - 3)^{2n}}{n \cdot 5^n}. \]

9. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n. \]

10. \[ \sum_{n=1}^{\infty} \frac{x^n}{e^{n^2}} \] (the root test is also a good option for this one).

So, why are we doing all of this? Well, functions are hard and often we cannot compute anything with them, but series are nice and computable. Let’s see an example of that.

11. Let’s go through these steps.

a) Find the integral of \( e^x \). Can you do the same thing for \( e^{x^2} \)?

b) It would be nice if we could rewrite everything in terms of power series and then compute things. But first some baby steps. If \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) then find \( \int f(x) \, dx \).

c) Let’s assume that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \). Use that to find power series representation of \( e^{x^2} \).

d) Use the previous two parts to find a power series representation for \( \int e^{x^2} \, dx \).

e) What is the integral \( \int_0^1 e^{x^2} \, dx \)?

Note we skipped one key step, namely finding the power series for the function \( e^x \). We will skip this for now but revisit in the very near future.
Let’s practice the idea of finding power series further

12. If \( \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \) then find a power series representation for the following functions.

a) \( \sin(x^2) \)

b) \( x^2 \sin(x) - x^2 \)

c) \( \frac{\sin(x)}{x} \)

13. Sometimes we can use derivatives to find power series.

a) If \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) then find a power series representation for \( f'(x) \).

b) Use that and the previous question to find a power series representation for \( \cos(x) \).

14. Let’s say you forgot the derivative of \( \sin(x) \). Use power series to show that the second derivative of \( \sin(x) \) is \( -\sin(x) \).

Congrats! You made it to the last question. As a reward you get to do this one last question.

15. Find the interval of convergence for the series \( \sum_{n=0}^{\infty} \left( \frac{x^2 + 1}{5} \right)^n \). Make sure you test the endpoints.

**Warning:** this one’s a little weird...