Study Goals:  • Study alternating series  
• Use the root test and ratio test

1. Fill in the blanks

An alternating series is one of the format \( \sum_{n=1}^{\infty} (-1)^n \) where \( b_n \) is any positive sequence.

An example of a alternating series is ____________

2. An alternating series \( \sum_{n=1}^{\infty} (-1)^n b_n \) is convergent if

a) \( b_n \) is ____________

b) \( b_n \) is ____________

c) \( b_n \) converges to _____

This called the ______________

3. Use the alternating series test to check the convergence of \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)?

4. a) Now that we can also have negative terms we have two different methods of convergence:

I Absolute Convergence: A series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent if ____________ is convergent.

II Conditional Convergence: A series \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent if ____________ is convergent but ____________ is divergent.

b) How do they compare?

I We know that ______________ implies ____________.

II However, ____________ does not imply ______________

c) Give an example to confirm that (II) does not always hold.
5. Alternating series can be sneaky and hide in plain sight. What can we say about the convergence of
\[ \sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n} \]?

6. The integral test gave us a means to approximate partial sums. However, this formula cannot be used for alternating series. For an alternating series \( \sum_{n=1}^{\infty} (-1)^n b_n \) we have the formula
\[ s_n < \ldots \]

7. a) Use the alternating series test to prove that the series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \) converges.

b) Using a calculator, find the partial sum \( s_6 = \sum_{n=0}^{6} \frac{(-1)^n}{n!} \) to four decimal places. What is the maximum value of \( |R_6| \)?

c) Soon we will prove that \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e} \). Compute \( \frac{1}{e} \) to four decimal places (with a calculator) to check that your answer agrees with this.

8. Consider the series
\[ 1 - \frac{1}{10^1} + \frac{1}{2} - \frac{1}{10^2} + \frac{1}{3} - \frac{1}{10^3} + \frac{1}{4} - \frac{1}{10^4} + \ldots \] (⋆)

a) Show that the series diverges. (Hint: What do you know about \( \sum_{n=1}^{\infty} \frac{1}{10^n} \)?)

b) Why doesn’t the Alternating Series Test apply to the series (⋆)?
We move on to the root test and ratio test

9. a) Ratio Test: Let \( r = \lim_{n \to \infty} \). If

I \( r < 1 \) then the series is ________________.

II \( r = 1 \) then ________________.

III \( r > 1 \) then the series is ________________.

b) Root Test: Let \( r = \lim_{n \to \infty} \). If

I \( r < 1 \) then the series is ________________.

II \( r = 1 \) then ________________.

III \( r > 1 \) then the series is ________________.

Remember the root test and ratio test only tell you the series is absolutely convergent!

10. a) Show that the root test is inconclusive when applied to \( \sum_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^{5n} \).

b) Use the divergence test to show that \( \sum_{n=1}^{\infty} \left( \frac{n}{n + 1} \right)^{5n} \) diverges. (Hint: Find \( \lim_{n \to \infty} \ln(a_n) \))

11. Use the root and/or ratio test to tell whether these series converge or diverge or the test is inconclusive:

a) \( \sum_{n=1}^{\infty} \frac{n + 3}{3^n} \)
12. Around 1910, the mathematician Ramanujan discovered the formula

\[ \frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4396^{4n}}. \]

William Gosper used this series in 1985 to compute \( \pi \) to 17 million digits.

(a) Verify that the series is convergent.

(b) How many correct decimal places of \( \pi \) do you get if you use just the first term of the series?
What if you use the first 2 terms? (Use a calculator.)

13. Challenge: Find a series which converges by the root test but not by the ratio test.