In this worksheet we will talk about the *Limit Comparison Test*.

1. Compute the following limits:
   
   a) \( \lim_{n \to \infty} \frac{n}{n^2} = \)
   
   b) \( \lim_{n \to \infty} \frac{2^n}{n^2} = \)
   
   c) \( \lim_{n \to \infty} \frac{n^3 + 1}{2n^3 - 5n} = \)
   
   d) \( \lim_{n \to \infty} \frac{3n^2 + 4 \ln(n)}{2n^2 - 5n + 7} = \)

2. What do you think your answer to the previous question tell you about the rate of growth of the top vs. the bottom? Use <, > or =.
   
   a) \( n \quad \square \quad n^2 \)
   
   b) \( 2^n \quad \square \quad n^2 \)
   
   c) \( n^3 + 1 \quad \square \quad 2n^3 - 5n \)
   
   d) \( 3n^2 + 4 \ln(n) \quad \square \quad 2n^2 - 5n + 7 \)

3. Think about what the comparative growth means for the relation between the series?

   Ok cool. Let's see a more concrete example:

4. 1. Find the following limit:
   
   \( \lim_{n \to \infty} \frac{n^3 + \ln(n) + 7n}{\frac{1}{n^2}} = \)

   2. What does you answer tell you about the rate of growth of top vs. bottom? \( \frac{n - 5}{n^3 + \ln(n) + 7n} \quad \square \quad \frac{1}{n^2} \)

   3. What does your answer tell you about the relation between the series
   
   \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{n - 5}{n^3 + \ln(n) + 7n} \)?

   4. Finally what does all of this tell me about convergence or divergence of \( \sum_{n=1}^{\infty} \frac{n - 5}{n^3 + \ln(n) + 7n} \)?
Now we will finally make this into an official theorem:

5. **Limit Comparison Test** If \( \{a_n\} \) and \( \{b_n\} \) are positive sequences and if

\[
\lim_{n \to \infty} \frac{a_n}{b_n} > 0
\]

is a finite number, then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are either both convergent or divergent.

Concretely \( \{a_n\} \) is the sequence that is provided to you in the question but \( \{b_n\} \) has to be chosen by you in a smart way. Note that in most cases \( \{a_n\} \) is in Let’s see how you can do this:

6. For each of the following problems first pick a sequence you want to compare the answer to (your \( b_n \)) and then use the limit comparison test to show that the sequence converges or diverges:

1. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \)

2. \( \sum_{n=1}^{\infty} \frac{n^4 + n^2 - n}{n^5 + 7n^4 + n^3 - 1} \)

3. \( \sum_{n=1}^{\infty} \frac{\sqrt{n^3 - n}}{n^2 + 4n - 1} \)

4. \( \sum_{n=1}^{\infty} \frac{\sqrt{(n - 5)^{10}}}{n^5 - 4n^4 + 4n - 5} \)

5. \( \sum_{n=1}^{\infty} \frac{n^2 + 4}{n^3 \ln(n) + n} \)

6. \( \sum_{n=1}^{\infty} \frac{2^n + 5n^2}{n! + 4n^6 - 27n^3} \)

7. \( \sum_{n=1}^{\infty} \frac{n^3 + 4n}{n^4 + 7} \)

7. Use the limit comparison test to determine the convergence/divergence of \( \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n}\right) \).