You will have 1 hour and 15 minutes to complete the following 15 problems. Use blank paper to write your answers, and don’t look at your notes!

Note: this exam is not 100% guaranteed to be free of errors — if you find an error, please email your TA.

Multiple Choice Questions

1. Let \( f(x) = \frac{\arctan(x)}{2} \). Find \( \frac{d^7}{dx^7} f(x) \big|_{x=0} \).

A. \( -\frac{7!}{30} \)
B. 0
C. \( -\frac{1}{30} \)
D. \( -\frac{7!}{14} \)
E. \( -\frac{1}{14} \)

**Solution:** Since the coefficient of the \( x^7 \) in the Maclaurin expansion of \( \frac{\arctan(x)}{2} \) is equal to \( \frac{f^7(0)}{7!} \), then we need only to find this coefficient, and then multiply it by 7! to get our answer. The Maclaurin expansion of \( \arctan(x) \) is given by

\[
\frac{1}{2} \arctan(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n + 1} x^{2n+1}
\]

Hence the coefficient of \( x^7 \) is given by \( \left( \frac{1}{2} \right) \frac{(-1)^3}{2(3) + 1} = -\frac{1}{14} \). Multiplying by 7! gives our final answer \(-\frac{7!}{14}\).

2. Find the Taylor expansion for \( \frac{2}{x^3} \) centered at \( a = 2 \)

A. \( \sum_{n=0}^{\infty} (x - 2)^n \)

B. \( \sum_{n=0}^{\infty} \frac{(-1)^n(n + 1)(n + 2)}{2^{n+3}} x^n \)

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C. \[ \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2^{n+3}}(x-2)^n \]

D. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2^{n+3}n!}(x-2)^n \]

E. \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+3}n!}(x-2)^n \]

**Solution:** We find the taylor expansion for \( \frac{2}{x^3} \) by differentiating and trying to look for a pattern in the coefficients. Since

\[
\begin{align*}
f(2) &= \frac{2}{2^3}, \quad f^{(1)}(2) = \frac{-2 \cdot 3}{2^4}, \quad f^{(2)}(2) = \frac{2 \cdot 3 \cdot 4}{2^5}, \\
f^{(3)}(2) &= \frac{-3 \cdot 4 \cdot 5}{2^6}
\end{align*}
\]

which suggests the pattern for the derivatives of this function at \( a = 2 \) is given by \( f^{(n)}(2) = \frac{(-1)^n(n+2)!}{2^{n+3}} \). So our \( n \)th coefficient is \( \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n(n+2)!}{2^{n+3}n!} = \frac{(-1)^n(n+1)(n+2)}{2^{n+3}} \) so our Taylor series is given by

\[
\frac{2}{x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)}{2^{n+3}}(x-2)^n
\]

3. Find the Taylor expansion for \( \frac{1}{1-x} \) centered at \( a = \frac{1}{2} \)

A. \[ \sum_{n=0}^{\infty} (x-1/2)^n \]

B. \[ \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n (x-\frac{1}{2})^n \]

C. \[ \sum_{n=0}^{\infty} 2^{n+1}(x-\frac{1}{2})^n \]

D. \[ \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-1} (x-\frac{1}{2})^n \]

E. \[ \sum_{n=0}^{\infty} 2^n(x-\frac{1}{2})^n \]
Solution: The Taylor expansion for \( \frac{1}{1-x} \) centered at \( a = \frac{1}{2} \) can be found first by finding the Maclaurin series for \( \frac{1}{1-(x+\frac{1}{2})} \). And the Maclaurin series for this second function can be determined more easily:

\[
\frac{1}{1-(x+\frac{1}{2})} = \frac{1}{\frac{1}{2} - x} = (\frac{1}{2}) \left( \frac{1}{1-(2x)} \right) = 2 \sum_{n=0}^{\infty} (2x)^n = \sum_{n=0}^{\infty} 2^{n+1} x^n
\]

\[
\Rightarrow \frac{1}{1-x} = \sum_{n=0}^{\infty} 2^{n+1} (x - 1/2)^n
\]

by making the substitution \( x + \frac{1}{2} \mapsto x, x \mapsto x - \frac{1}{2} \).

For the next 3 questions specify whether the given series absolutely converges, conditionally converges or diverges.

4. \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)} \)

   A. Converges Absolutely
   B. Converges Conditionally
   C. Diverges

Solution: The series is alternating so we use the alternating series test.

1) \( \frac{1}{n \ln(n)} \) is positive

2) \( \ln(n)n < \ln(n+1)(n+1) \Rightarrow \frac{1}{\ln(n+1)(n+1)} < \frac{1}{\ln(n)n} \), so it is decreasing.

3) \( \lim_{n \to \infty} \frac{1}{n \ln(n)} = 0. \)

So, by the alternating series test the series converges.

However the series does not absolutely converge i.e. \( \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \) diverges. For that we use the integral test. The function \( \frac{1}{x \ln(x)} \) is positive, continuous and decreasing. So,
we will check the behavior of the improper integral instead. Using the substitution $u = \ln(x)$ gives us

$$
\int_{2}^{\infty} \frac{1}{x \ln(x)} \, dx = \int_{\ln(2)}^{\infty} \frac{1}{u} \, du
$$

which diverges by the $p$-test.

5. $\sum_{n=2}^{\infty} \frac{(n-2)^n(3^n + 1)}{(2n+1)^n}$

A. Converges Absolutely
B. Converges Conditionally
C. Diverges

**Solution:** We use the root test.

$$
\lim_{n \to \infty} \sqrt[n]{\frac{(n-2)^n(3^n + 1)}{(2n+1)^n}} = \lim_{n \to \infty} \frac{(n-2)3}{2n+1} = \frac{3}{2} > 1
$$

So, by the root test this series diverges.

6. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^2}$

A. Converges Absolutely
B. Converges Conditionally
C. Diverges

**Solution:** Notice that $|\cos(n\pi)| = 1$ for any $n$. So $\left| \frac{\cos(\pi n)}{n^2} \right| = \frac{1}{n^2}$. But this converges by the $p$ test, which means the series converges absolutely.

7. Suppose I have a positive sequence $a_n$ where $\sum_{n=0}^{\infty} a_n$ converges. What can I say about the series $\sum_{n=0}^{\infty} (-1)^n a_n$?

A. $\sum_{n=0}^{\infty} (-1)^n a_n$ converges conditionally.
B. $\sum_{n=0}^{\infty} (-1)^n a_n$ diverges.
C. $\sum_{n=0}^{\infty} (-1)^n a_n$ converges absolutely.

D. Both A. and C.

E. There’s not enough information.

**Solution:** If you take the absolute value of the terms of $\sum_{n=0}^{\infty} (-1)^n a_n$, you get $\sum_{n=0}^{\infty} a_n$, which we are told converges. So, the series converges absolutely.

8. Evaluate the limit $\lim_{x \to 0} \frac{e^{-x^3} + x^3 - 1}{x^6}$

   A. $\frac{1}{2}$
   B. $\infty$
   C. 1
   D. 0
   E. $\frac{1}{6}$

**Solution:** Take the power series representation of $e^x$ and substitute in $-x^3$ to get

$$e^{-x^3} = 1 - x^3 + x^6 / 2! - x^9 / 3! + \ldots$$

When I add the $x^3$ and subtract $-1$, this exactly cancels the first two terms. Dividing then by $x^6$ gives

$$(e^{-x^3} + x^3 - 1) / x^6 = 1/2! - x^3 / 3! + \ldots$$

Taking the limit, all terms with a positive power of $x$ will go to zero, leaving just the $1/2$.

9. Find the radius of convergence of the following power series

$$\sum_{n=1}^{\infty} \frac{(2n)! (x - 2)^n}{2^n (n!)^2}$$

   A. $\frac{1}{2}$
   B. 2
   C. 1
   D. 0
   E. $\infty$
Solution: Using the ratio test, we will get
\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(2n+2)! (x - 2)^{n+1}}{2^{n+1} ((n+1)!)^2} \cdot \frac{2^n (n!)^2}{(2n)! (x - 2)^n} \right|
\]
After grouping and cancelling terms
\[
= \left| \frac{1}{2} \frac{(2n+2)(2n+1)}{(n+1)^2} (x - 2) \right|
\]
Coming from simplifying and grouping the powers of 2, the factorial terms, and, and the powers of \((x - 2)\), respectively. Taking the limit as \(n \to \infty\):
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{2^2}{2} |x - 2| < 1
\]
And so
\[
|x - 2| < 1/2
\]
This means the radius of convergence is 1/2.

10. Find the Taylor series at \(x = -1\) of the following function:
\[f(x) = 2x^3 + x^2 - 4x + 1\]

A. \(4 - 5(x - 1)^2 + 2(x - 1)^3\)
B. \(4 - 5(x + 1) + 2(x + 1)^2\)
C. \(4 - 10(x + 1)^2 + 12(x + 1)^3\)
D. \(4(x - 1) + 7(x - 1)^2 + 2(x - 1)^3\)
E. \(4 - 5(x + 1)^2 + 2(x + 1)^3\)

Solution: Starting with \(f(x) = 2x^3 + x^2 - 4x + 1\), we take derivatives and find the values at \(-1\):
\[
f(-1) = 4
\]
\[
f'(-1) = 0
\]
\[
f''(-1) = -10
\]
\[
f'''(-1) = 12
\]
And then plug in to the formula for the Taylor series to obtain the correct solution:
\[
f(x) = 4 + 0(x + 1) - \frac{10}{2!} (x + 1)^2 + \frac{12}{3!} (x + 1)^3
\]
11. Give the Maclaurin series for the function $x^2 e^{4x^3}$. Make sure that your answer is in power series form.

**Solution:** You do not want to try to differentiating this function. It can get messy very quickly, and it’s hard to find a pattern in the coefficients.

Your best option is to use the fact that you know the Maclaurin series of $e^x$, and manipulate this to obtain the Maclaurin series of your desired function. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$x^2 e^{4x^3} = x^2 \left( \sum_{n=0}^{\infty} \frac{(4x^3)^n}{n!} \right) = x^2 \left( \sum_{n=0}^{\infty} \frac{4^n}{n!} x^{3n} \right)$$

$$= \sum_{n=0}^{\infty} \frac{4^n}{n!} x^{3n+2}$$

12. (a) Give the statement of the alternating series test.

**Solution:** Given a series $\sum_{n=1}^{\infty} (-1)^n b_n$, such that $b_n$ is positive, decreasing, and $\lim_{n \to \infty} b_n = 0$, we can conclude that the series is convergent. The test does not determine if a series is absolutely convergent, but it does not exclude this possibility.

(b) Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{3n^2 + 13}$ converge or diverge? Justify why.

**Solution:** Using the alternating series test, we can see that the sequence $\{b_n\} = \{\frac{1}{3n^2 + 13}\}$ is positive, decreasing, and $\lim_{n \to \infty} \frac{1}{3n^2 + 13} = 0$, so this series converges.

(c) If convergent, is the series absolutely convergent or conditionally convergent? Justify why.

**Solution:** This series converges absolutely. Since the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{3n^2 + 12}$ is comparable to $\sum_{n=2}^{\infty} \frac{1}{n^2}$ from an application of the limit comparison test, and this second series is convergent (by the $p$-test) then $\sum_{n=2}^{\infty} \frac{1}{3n^2 + 12}$ converges. Hence $\sum_{n=2}^{\infty} \frac{(-1)^n}{3n^2 + 13}$ converges absolutely.
13. Find the interval of convergence of the following power series:

\[ \sum_{n=2}^{\infty} \frac{e^n(x-4)^{3n}}{\ln n} \]

**Solution:** We will use the ratio test for \( a_n = \frac{e^n(x-4)^{3n}}{\ln n} \):

\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{e^{n+1}(x-4)^{3(n+1)}}{\ln(n+1)} \cdot \frac{\ln n}{e^n(x-4)^{3n}} \right| = \left| \frac{e^{n+1}(x-4)^{3(n+1)} \ln(n+1)}{e^n(x-4)^{3n} \ln n} \right| \]

Simplifying and taking the limit as \( n \to \infty \),

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| e(x-4)^3 \frac{\ln(n+1)}{\ln n} \right| = e|x-4|^3 \]

because the limit of \( \frac{\ln(n+1)}{\ln n} \) is 1 as \( n \to \infty \).

To find the interval where it converges absolutely, we find for what \( x \) this is less than 1:

\[ e|x-4|^3 < 1 \]
\[ |x-4|^3 < \frac{1}{e} \]
\[ |x-4| < \frac{1}{e^{1/3}} \]

Finally, we check the endpoints. At \( x = 4 - 1/e^{1/3} \), the original sum becomes

\[ \sum_{n=2}^{\infty} \frac{e^n(-1/e^{1/3})^{3n}}{\ln n} = \sum_{n=2}^{\infty} \frac{e^n((-1/e^{1/3})^{n}}{\ln n} = \sum_{n=2}^{\infty} \frac{e^n(-1/e)^n}{\ln n} \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \]

This converges by the alternating series test. For \( x = 4 + 1/e^{1/3} \), we get

\[ \sum_{n=2}^{\infty} \frac{e^n(1/e)^n}{\ln n} \sum_{n=2}^{\infty} \frac{1}{\ln n} \]

which diverges by direct comparison with \( \sum 1/n \).

14. a) Write down a series for \( \int_0^1 x^2 \sin(x^2) \, dx \)

b) What is the least number of terms we have to add to approximate this series with an error of less than \( \frac{1}{1000} \)?
Solution: a) We use the fact that

\[ \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \]

We get

\[ \int_0^1 x^2 \sin(x^2) \, dx = \int_0^1 x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \, dx = \int_0^1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{(2n+1)!} \, dx = \]

\[ \left. \frac{\xi^{4n+5}}{(4n+5)(2n+1)!} \right|_0^1 = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+5)(2n+1)!} \]

b) For that we use the fact that the series is alternating. We have \( |R_n| \leq b_{n+1} \), so we need to find an \( n \) such that \( b_{n+1} < \frac{1}{1000} \). For that we just try numbers.

\[ b_{0+1} = \frac{1}{(4(1) + 5)(2(1) + 1)!} = \frac{1}{(9)(6)} = \frac{1}{54} > \frac{1}{1000} \]

\[ b_{1+1} = \frac{1}{(4(2) + 5)(2(2) + 1)!} = \frac{1}{(13)(120)} < \frac{1}{1000} \]

So, the answer is \( n = 1 \). So, we have to add at least 2 terms.

15. Let \( f(x) = \sqrt[3]{x^7} - x^2 + 2x \).

a) Find the degree 2 Taylor polynomial centered at 8.

b) Find the maximum error of this approximation of the Taylor series on the interval \([1, 10]\)

Solution: a) We evaluate the derivative of the function at 8.

\[ f(8) = \sqrt[3]{8^7} - 8^2 + 2(8) = 80 \]

\[ f'(8) = \frac{7}{3} \sqrt[3]{(8)^4} - 2(8) + 2 = \frac{70}{3} \]

\[ f''(8) = \frac{28}{9} \sqrt[3]{8} - 2 = \frac{38}{9} \]

So the answer is

\[ T_2(x) = 80 + \frac{70}{3} (x - 8) + \frac{38}{9} \frac{(x - 8)^2}{2} \]
b) For this part we use Taylor’s inequality. First we find the third derivative.

\[ f^{(3)}(x) = \frac{28}{27} \frac{1}{\sqrt{x^2}} \]

This function is decreasing on the given interval, so it will attain its maximum value at the point 1, so \( M = \frac{28}{27} \frac{1}{\sqrt{1^2}} = \frac{28}{27} \). Also the function \(|x - 8|\) attains its maximum value at the point 1.

By the Taylor’s inequality the maximum error is

\[ R_2(x) \leq \frac{28}{3!} |1 - 8|^3 = \frac{28}{27(3!)} 7^3 \]