Math 221/AL1 Exam III

UIUC, December 5, 2013

1. (12 points, 3/3/6) This question concerns the definite integral

\[ \int_{1}^{3} (2x + 1) \, dx \]

(a) Sketch the area represented by the integral and compute it using geometry.
(b) Solve the integral using the fundamental theorem of calculus.
(c) Compute the integral by computing the limit of the Riemannian sums as the partitions go to \( \infty \).

2. (20 points) Compute the indefinite integral.

(a) \( \int e^{4x} \, dx \)
(b) \( \int x\sqrt{x-1} \, dx \)
(c) \( \int \frac{1}{16 + x^2} \, dx \)
(d) \( \int \sec^2 \theta \tan^2 \theta \, d\theta \)
(e) \( \int \frac{2x + 4}{\sqrt{1 - x^2}} \, dx \)

3. (12 points, 4/4/4) The shaded region is bounded by the curves \( y = x^2 + x + 1 \) and \( y = 2x^2 - 1 \)

(a) Give an integral (do not evaluate) for the area of the region.
(b) Give an integral (do not evaluate) for the volume obtained if this region is rotated about the line \( y = 5 \).
(c) Give an integral (do not evaluate) for the volume obtained if this region is rotated about the line \( x = 6 \).
4. (8 points) Write an integral (you do not need to evaluate it) for the volume of a piece of a right circular cone whose base radius \( b = 10 \) and top radius \( a = 6 \) and overall height is 5.

5. (8 points) Give an integral (you do not need to compute it) for the volume of a “rugby ball” if it has length 300 mm and circumference 600 mm and it is formed by rotating an ellipse about its major axis. Recall that the equation of an ellipse is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

6. (6 points) This question concerns the function \( g(x) = \int_0^x f(t) \, dt \) if the graph of the continuous function \( f(x) \) from 0 to 5 is

\[
\begin{align*}
y &= f(x)
\end{align*}
\]

Estimate the following to one decimal place.

(a) Where are the critical points of \( g(x) \)?

(b) Where is \( g(x) \) increasing/decreasing?

(c) For what value \( c \) is \( g(c) \) maximal?
7. (6 points) Suppose you know the following
\[ \int_1^3 f(x) \, dx = 4 \quad \int_3^9 f(x) \, dx = 6 \]

Compute the following integrals.
(a) \[ \int_1^9 f(x) \, dx \]
(b) \[ \int_3^1 2f(x) \, dx \]
(c) \[ \int_1^3 f(3x) \, dx \]

8. (8 points) (Short answer)
The following is a graph of \( f(x) = \frac{1}{x} \) with some shaded rectangles.

(a) Explain how the diagram leads to the equation
\[ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} < \ln 2 \]

(b) Explain why
\[ \lim_{n \to \infty} \left( \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+n} \right) = \ln 2 \]