Final Exam:

Time: 7:00-10:00 PM on Tuesday, December 15th, 2015
Place: 314 Altgeld Hall
Sections: 2.1-2.3, 2.5-2.8, 3.1-3.2, 3.4-3.9, 4.1-4.3, 4.7, 4.9, 5.1-5.5, 6.1-6.4

• Your exam is three hours long. So make sure you prepare well and rest so that you are ready for a very long exam session. Study hard!!!

• Don’t forget to bring your ID with you as there will be an ID check.

• This exam is 30% of your final grade and so it’s one third of your whole grade and twice as much as any midterm. Don’t miss this opportunity to get a lot of points.

• I recommend going over the list of materials mentioned in this sheet, then identifying which topics you are not comfortable with and working on the worksheets related to those topics.
Review (Things you should definitely know):

First Part:

1. **Functions:** This is mainly background material. Remember what a function is and basic facts about functions. In particular, you should know what polynomials, trigonometric, exponential and logarithmic functions are.

2. **Limits:** Remember how to solve limits. Most of them can be solved by using one of the following methods:

   I) Plugging in and calculating. Example: \( \lim_{x \to -\infty} \frac{e^x - 3}{e^x - 1} \)

   II) Factorization. Example: \( \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} \)

   III) Multiplying by conjugates. Example: \( \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \)

   IV) Dividing top and bottom of a fraction by the highest power (for infinite limits of rational functions) Example: \( \lim_{x \to +\infty} \frac{x^2 - 2}{3x^2 + 5} \)

   V) Using the Squeeze Theorem. Example: \( \lim_{x \to +\infty} \frac{\sin^2(x)}{x} \)

   VI) Using equalities like \( a^2 - b^2 = (a-b)(a+b) \). Example: \( \lim_{h \to 0} \frac{1}{(x+h)^2} - \frac{1}{x^2} \)

3. **Continuity:** Remember all basic functions are continuous. In particular, polynomials, exponential, logarithmic and trig functions. You don’t have to memorize the precise definition of continuity but a good understanding of it can be helpful.

4. **IVT:** Remember the precise conditions of IVT. Make sure you state that your function is continuous and then check the other conditions.

5. **Asymptotes:** A function \( f(x) \) can have two kinds of asymptotes: Horizontal and Vertical.

   Horizontal Find the limits \( \lim_{x \to +\infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). If any of them converge to a number \( c \) then the line \( y = c \) is a horizontal asymptote. There can be at most two of them.
Vertical First identify all suspicious points \( a \), like roots of denominators. Then calculate the limit to \( a \). If the limit \( \lim_{x \to a^+} f(x) = \pm \infty \) or \( \lim_{x \to a^-} f(x) = \infty \) then \( x = a \) is a vertical asymptote.

6. **Limit definition of Derivative**: There are two common ways to define derivative of \( f(x) \) at the point \( a \) using limits:

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \text{and} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}
\]

Try to choose the one that results in the easiest algebra.

7. **Derivative Rules**: We have the following derivative rules. For two differentiable function \( f(x) \) and \( g(x) \) and constant \( c \),

- **Addition Rule**: \( \frac{d}{dx}(f \pm g) = \frac{d}{dx}f \pm \frac{d}{dx}g \)
- **Product Rule**: \( \frac{d}{dx}(fg) = (\frac{d}{dx}f)g + f(\frac{d}{dx}g) \)
- **Quotient Rule**: \( \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{d}{dx}(f)g - f(\frac{d}{dx}g)}{g^2} \)
- **Coefficient Rule**: \( \frac{d}{dx}(cf) = c\frac{d}{dx}(f) \)
- **Power Rule**: \( \frac{d}{dx}(x^n) = nx^{n-1} \)

8. **Standard Derivatives**: Remember the derivatives of standard functions:

- **Power Functions**: \( \frac{d}{dx}(x^n) = nx^{n-1} \)
- **Trigonometric Functions**: \( \frac{d}{dx}(\sin(x)) = \cos(x) \), \( \frac{d}{dx}(\cos(x)) = -\sin(x) \), \( \frac{d}{dx}(\tan(x)) = \tan^2(x) + 1 = \sec^2(x) \), \( \frac{d}{dx}(\cot(x)) = -\cot^2(x) - 1 = -\csc^2(x) \)
Exponential Functions: 
\[ \frac{d}{dx}(a^x) = \ln(a)a^x \]

Logarithmic Functions: 
\[ \frac{d}{dx}(\log_a(x)) = \frac{1}{\ln(a)x} \]

Second Part:

9. **Chain Rule:** If \( f \) and \( g \) are differentiable then we have
\[ (g \circ f)'(x) = f'(x)g'(f(x)) \]

10. **Sketch Graphs:** Extract information from functions and sketch them
    I) **Basic Info:** Find domain, roots (zero) and symmetries of the function
    II) **Asymptotes:** Find the vertical and horizontal asymptotes of functions
    III) **First Derivative:** Find critical points, local and global extrema, intervals of increase and decrease
    IV) **Second Derivative:** Find inflection points, intervals of concavity up and down
    V) **Sketch:** Sketch the function using all these information

11. **V Theorems:** Learn the applications of the five important theorems:
    I) **Intermediate Value Theorem:** To show the existence of roots.
    II) **Mean Value Theorem:** Show \( f' \) is equal to a certain value
    III) **Rolle’s Theorem:** Mostly used to show a function has one root.
    IV) **Fermat’s Theorem:** To find local max and min
    V) **Extreme Value Theorem:** To find global max and min

12. Two new methods to find limits:
    I) **L’Hospitals Rule:** Use L’Hospitals rule to calculate limits of fractions.
    II) **Ln-method:** Use \( \ln \) method to find limits of exponential functions \((f(x)^{g(x)})\) (You normally also need L’Hospitals rule)
13. **Implicit Differentiation:** Be careful with variables and use chain rule correctly.

14. **Related Rates and Optimization:** There are two steps:
   
   I) Find correct formula in terms of given variables. It is very important to identify correct relations.
   
   II) Take the derivative and plug in values or find roots

Third Part:

15. **Antiderivatives:** Make sure you know the antiderivatives of standard functions, such as polynomials, exponential functions, trig function and inverse trig functions.

16. **Riemann Sums:** Remember the formula for Riemann sum:

   \[
   \text{Area} = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + \frac{(b-a)i}{n})(\frac{b-a}{n}).
   \]

   You will need the following formulas to solve Riemann sum questions:

   1) \( \sum_{i=1}^{n} 1 = n \)
   2) \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)
   3) \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)
   4) \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \)

17. **Fundamental Theorem of Calculus:** There are two theorems (FTC I and II):

   I) FTC I: The function \( g(x) = \int_{a}^{x} f(t) \, dt \) is differentiable and
   
   \[ g'(x) = f(x). \]

   II) FTC II: If the function \( F(x) \) is the anti-derivative of \( f(x) \) then we have:
   
   \[ \int_{a}^{b} f(t) \, dt = F(b) - F(a). \]

18. **Properties of Integrals:**
I)
\[ \int_{a}^{b} (f(t) \pm g(t)) \, dt = \int_{a}^{b} f(t) \, dt \pm \int_{a}^{b} g(t) \, dt \]

II)
\[ \int_{a}^{b} (c f(t)) \, dt = c \int_{a}^{b} f(t) \, dt \]

III)
\[ \int_{a}^{b} f(t) \, dt = - \int_{b}^{a} f(t) \, dt \]

IV)
\[ \int_{a}^{b} f(t) \, dt = \int_{a}^{c} f(t) \, dt + \int_{c}^{b} f(t) \, dt \]

V) If \( f(x) \geq 0 \) then \( \int_{a}^{b} f(t) \, dt \geq 0 \)

VI) If \( f(x) \geq g(x) \) then \( \int_{a}^{b} f(t) \, dt \geq \int_{a}^{b} g(t) \, dt \)

VII) If \( m \leq f(x) \leq M \) then \( m(b-a) \leq \int_{a}^{b} f(t) \, dt \leq M(b-a) \)

VIII) If \( f(x) \) is even i.e. \( f(-x) = f(x) \), then \( \int_{-b}^{b} f(t) \, dt = 2 \int_{0}^{b} f(t) \, dt \)

(notice the endpoints are \(-b\) and \(b\)).

IX) If \( f(x) \) is odd i.e. \( f(-x) = -f(x) \), then \( \int_{-b}^{b} f(t) \, dt = 0 \) (again notice the endpoints are \(-b\) and \(b\)).

19. **U-Substitution:** Remember how to use \( u \)-substitution for harder integrals. Make sure you practice as it can be difficult to choose correct substitution.

20. **Finding Areas and Volume:** There are three different kinds of questions regarding:

I) **Finding Areas:** In order to find the area between two graphs \( f(x) \) and \( g(x) \), you have to solve the equation \( f(x) = g(x) \) in
order to identify the points where $f(x)$ and $g(x)$ are equal. You should also always have a rough sketch of the graphs to get an idea of how what the bounds of the area you want to find are. After you have gathered all this information the area is

$$\int_a^b (f(t) - g(t)) \, dt$$

for the parts where $f(x) > g(x)$ and $\int_a^b (g(t) - f(t)) \, dt$ for the parts where $g(x) > f(x)$.

In short term, the formula for the area between $f(x)$ and $g(x)$ in the interval $a$ and $b$ is:

$$\int_a^b |f(t) - g(t)| \, dt$$

II) Finding Volume of Cross-Sections: In order to find the volume of a shape which cross-section perpendicular to the $x$-axis is a specific geometric shape (like a square, an equilateral triangle, a circle or a semi-circle, ... ), we have to do two things. First we go through the first step to identify the area and then we plug in the calculated side length into the formula of the area. In a formula this can be given as:

$$\int_a^b A(|f(t) - g(t)|) \, dt$$

where $A(x)$ is the formula of the area of the cross-section. (Note that in the particular case where the cross-sections are circles the value $|f(t) - g(t)|$ is the diameter and not the radius and so has to be divided by two and then plugged into the formula for the area).

III) The Volume of Rotation: The volume of rotation can be calculated using two methods: The Disk (Washer) method and the Shell method. They come from different geometric methods how to split up a cylinder: into many shells or into many discs.

i. Disc Method: We find the area of an annulus: $\pi(R^2 - r^2)$ (where $R$ and $r$ are the radii of the annulus) and add them up over the given interval:

$$\int_a^b \pi(R(t)^2 - r(t)^2) \, dt$$
ii. **Shell Method:** We find the surface area of a shell: \(2\pi rh\) and then add them up over the given interval:

\[
\int_a^b 2\pi r(t)h(t) \, dt
\]

As you see the formulas can be memorized very fast. The main issue is to use geometry to identify the radius and or height correctly which can be confusing. Therefore it is very important to practice a lot.
Other things to remember:

1. Remember \( \lim_{x \to -\infty} e^x = \lim_{x \to +\infty} \frac{1}{e^x} = 0 \)

2. Remember to check the points you think are vertical asymptotes by actually taking limits.

3. Remember, when checking for vertical asymptotes, that zeroes of denominators are actually asymptotes and not just holes

4. The most common IVT questions are either ”show \( f \) has a root” or ”show these two functions \( f \) and \( g \) are equal at some point”.

5. Remember, when taking limits of products, you can make products into fractions and use L’Hospital’s rule

6. Remember to not mix up L’Hospital’s rule with the quotient rule

7. Related Rates and optimization rely on your experience so practice hard

8. DON’T use theorems when the question asks you to use the limit definition of derivatives or the Riemann sum definition of integrals.

9. Remember FTC I does not involve finding the antiderivative of the function. Understand the difference between FTC I and FTC II.

10. Remember the \( c \) when calculating indefinite integrals.

11. Remember to change the bounds of definite integrals, when you use \( u \)-substitutions.

12. Remember to write your answer in terms of the initial variable in case you used substitution for indefinite integrals.

13. (VERY IMPORTANT) Remember the difference between integral and area.

14. One of the trickier questions is to recognize when to use shell method and when to use disc method (when you want to find the volume). So, it is important to go over some examples to see how things work out.
15. Make sure you go over the examples in section 6.4 to understand how you can use definite integrals to compute work.

16. Remember the integrals of even and odd functions over symmetric intervals.

17. Remember the formula of areas of squares, triangles and circles.

18. Remember standard facts about algebra, for example:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}
\]

19. Remember trig identities and important angles

20. Remember exponentials and log identities, for example:

\[
\log(ab) = \log(a) + \log(b)
\]

21. More generally, be careful when using algebra, particularly when dealing with fractions

22. Write carefully and complete answers

Good Luck with your finals! I hope you do well !!!