

On Triples, Operads, and Generalized Homogeneous Functors

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Abstract

We study the splitting of the Goodwillie towers of functors in various settings. In particular, we produce splitting criteria for functors $F : \mathcal{A} \rightarrow M_A$ from a pointed category with coproducts to A -modules in terms of differentials of F . Here A is a commutative S -algebra. We specialize to the case when \mathcal{A} is the category of \mathfrak{a} -algebras for an operad \mathfrak{a} and F is the forgetful functor, and derive milder splitting conditions in terms of the derivative of F . In addition, we describe how triples induce operads, and prove that, roughly speaking, a triple T is naturally equivalent to the product of its Goodwillie layers if and only if it is an algebra over its induced operad.

Key words: spectra with additional structure, Goodwillie Calculus, algebras over operads

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1 Introduction

One of the central results in Homological Algebra is the Hochschild-Kostant-Rosenberg (HKR) theorem (e.g. see Chapter 3 of [16]), which states that for a smooth algebra map $k \rightarrow A$, the Hochschild homology coincides with the differential forms:

$$HH_*(A) = \Omega_{A|k}^*$$

This is a ‘splitting’ result in its essence, as it is a consequence of the collapsing the André-Quillen fundamental spectral sequence for HH by constructing certain section maps.

The splitting nature of the HKR theorem is more apparent in its topological analogue which we developed in [20]. There we work in the symmetric monoidal category of S -modules invented by Elmendorf et al in [7], and prove that under suitable conditions, the Topological Hochschild homology of a connective commutative S -algebra A decomposes:

$$\mathbb{P}_A \Sigma T A Q(A) \xrightarrow{\cong} T H H(A),$$

where $\mathbb{P}_A(X) = \bigoplus X^{\wedge n} / \Sigma_n$ is the symmetric algebra triple, and $\Sigma TAQ(A)$ is the suspension of the Topological André-Quillen homology.

The proof of this result is, in essence, a two step process. First, we develop splitting criteria for commutative S -algebras, then show that if the commutative S -algebra A is sufficiently ‘nice’ (or more precisely smooth in terminology of [20]), then the commutative S -algebra $THH(A)$ satisfies these splitting criteria. This served as one of the main motivations for the present work, as it gave rise to natural questions. Is the *commutative* algebra structure special, or can the splitting results be extended to algebras of other types? More generally, when can we decompose functors? Addressing the ambiguity of the last question, we will explain shortly in what sense it is a generalization of the first one, but now we digress a little to make the notion of ‘decomposing functors’ more precise, which leads to an explanation of the terminology used in the title.

In a series of papers [11], [12] and [13], Tom Goodwillie developed a theory (the calculus of homotopy functors), which has since become an important tool in homotopy theory. The central ingredient of his theory is a construction of a natural inverse limit system of functors

$$\cdots \rightarrow P_n F \rightarrow P_{n-1} F \rightarrow \cdots \rightarrow P_1 F$$

for a suitable functor F . This system, which is usually referred to as Goodwillie or Taylor tower of F , plays a role analogous to that of Taylor series in real variable calculus. In particular, often, though not always, the system $\{P_n F(X)\}$ converges to $F(X)$. However, unlike functions of real variables, the limit of the Taylor tower of functors does not necessarily decompose into the product of the fibers (or layers) $D_n F(X) \stackrel{def}{=} fiber[P_n F(X) \rightarrow P_{n-1} F(X)]$ (or equivalently, the Taylor tower does not necessarily split). For the cases when it does, we introduce a new terminology.

To do so, first recall that by definition (due to T. Goodwillie), the functor G is n -homogeneous if the Taylor polynomials $P_i G$ are contractible for all $i < n$ and $P_i G \simeq G$ for $i \geq n$. In particular, the layers $D_n F$ are n -homogeneous. Thus, when the Goodwillie tower splits, its homotopy limit is equivalent to a product of homogeneous functors. Keeping in mind that *generalized* Eilenberg-MacLane spaces are precisely the spaces which decompose into products of fibers of their Postnikov systems, we present our definition.

Definition 1.1. Functor F is called a *generalized* homogeneous functor if it is equivalent to a product of n -homogeneous functors (not necessarily for the same n).

Functors with splitting Taylor towers provide the principal class of examples of generalized homogeneous functors, since for such functors, the functor $P_\infty F \stackrel{def}{=} \text{holim } P_n F$ is equivalent to the product of the (n -homogeneous) fibers of the tower, and thus is generalized homogeneous.

We return to our motivational example of the HKR theorem developed in [20]. There, as a key step, we construct a tower of functors that approximates the forgetful functor U from the category of non-unital commutative A -algebras to A -modules, and discuss conditions under which that tower splits. Here A is a cofibrant commutative S -algebra and ‘approximates’ means that under suitable conditions, the homotopy limit of the tower is equivalent to the functor. Not unexpectedly, this tower is equivalent to the Taylor tower of U , though we have intentionally avoided the Goodwillie Calculus language in [20] to increase its accessibility.

In this paper, we consider the conditions under which the forgetful functor from the category of algebras of other types (such as associative, or Lie) is generalized homogeneous at a given object. To make this statement precise we utilize the formalism provided by the language of operads. In other words, for a fixed operad \mathbf{a} , we consider the forgetful functor $U_{\mathbf{a}} : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$, where $\mathcal{C}_{\mathbf{a}}$ is the category of \mathbf{a} -algebras and \mathcal{M}_A - of A modules, and explore the question of the splitting of the Goodwillie tower of $U_{\mathbf{a}}$ evaluated at a fixed \mathbf{a} -algebra X .

More generally still, we discuss the splitting of the Goodwillie tower of a functor $F : \mathcal{A} \rightarrow \mathcal{M}_A$ from *any* pointed category with coproducts. In fact, we produce conditions which are both necessary and sufficient for such towers to split. To explain these, we recall a generalization of the notion of derivative, which is introduced in Section 5 of [14], and is also briefly described here, in Sections 2 and 3. First, the derivative itself is simply the functor $P_1F = D_1F$, which is linear and comes equipped with a natural (derivative) map $F(X) \rightarrow D_1F(X)$. As with functions of real variables, one can define a notion of a directional derivative in Goodwillie Calculus to extend this. In other words, we can take the derivative of the functor F at an object Y in direction of an object X (see Definition 2.6). The derivative $D_1F(X)$ is simply the directional derivative at the base point $*$ in direction of X .

An additional piece of notation will allow us to state one of our main results in a rather compact form. Define \mathcal{X} to be the full subcategory of \mathcal{A} whose objects are $\vee_n X$ for all $n \geq 0$ (with $\vee_0 X = *$), and denote by $F|_{\mathcal{X}}$ the restriction of F to \mathcal{X} .

Theorem 1.2. *The functor $P_{\infty}(F|_{\mathcal{X}})$ is generalized homogeneous if and only if the natural (derivative) map from $F(X \vee Y)$ to the directional derivative of $F|_{\mathcal{X}}$ at Y in direction of X has a section for all $Y \in \mathcal{X}$.*

In particular, this theorem provides criteria under which the forgetful functor $U_{\mathbf{a}}$ evaluated at an algebra X (and hence the algebra X itself) decomposes (if the homotopy limit of the Taylor tower of $U_{\mathbf{a}}$ is equivalent to $U_{\mathbf{a}}$ itself). This approach however ignores the additional structure of an algebra that the objects under discussion possess. The special case of commutative algebras mentioned above though, suggests that it is of critical importance. Indeed, in [20], we showed that the Goodwillie tower of the forgetful functor U from

the category of commutative A -algebras to A -modules splits if the natural derivative map $U(X) \rightarrow D_1U(X)$ has a section. In other words, to produce a decomposition for commutative algebras, we required that in terminology of Theorem 1.2, the directional derivative be equipped with a section only for $Y = *$, as opposed to $Y = \vee_n X$ for all n .

We are able to get a similar result for algebras over other operads as well. We do restrict however to operads \mathbf{a} with $\mathbf{a}(1)$ equivalent to the unit A of the symmetric monoidal category \mathcal{M}_A . Note that this is not a very restrictive assumption as most operads naturally occurring in literature satisfy it.

Theorem 1.3. *Let \mathbf{a} be an operad with $\mathbf{a}(1)$ equivalent to A , and let C be an \mathbf{a} -algebra such that the natural derivative map $U_{\mathbf{a}}(C) \rightarrow D_1U_{\mathbf{a}}(C)$ has a section in the category of A -modules. Then*

$$\operatorname{holim}_n P_n U_{\mathbf{a}}(C) \simeq \prod \mathbf{a}(n) \wedge_{\Sigma_n} [D_1U_{\mathbf{a}}(C)]^{\wedge n}.$$

Note that as a trade off for relaxing the splitting criteria of Theorem 1.2, we claim that the Goodwillie tower of $U_{\mathbf{a}}$ splits only at C and not all finite multiples of C .

The special case of the commutative algebra operad \mathbf{e}_{∞} implies the Hochschild-Kostant-Rosenberg theorem. Another immediate application of this result recovers the theorem of Leray on the structure of commutative quasi Hopf algebras, and consequently the Poincaré-Birkhoff-Witt theorem.

Yet another special case of Theorem 1.2 is obtained by considering only those functors which have the additional structure of a triple. In fact, a splitting result on a triple T would be a natural culmination of the two main theorems presented so far, since for all X , $T(X)$ is equipped with a structure of an algebra. Complications arise due to the fact that not every triple is induced by an operad, and thus $T(X)$ may not be an algebra over a triple. Hence Theorem 1.3 may not be applicable.

To remedy this problem, we present a construction which is of interest on its own. It is based on a simple observation that if $T_{\mathbf{a}}$ is the triple associated with the operad \mathbf{a} in the symmetric monoidal category \mathcal{M}_S of S -modules, then we can recover the n 'th space $\mathbf{a}(n)$ of the operad by multilinearizing the functor $cr_n T_{\mathbf{a}}$ at each of its n variable, where cr_n is the n 'th cross effect (see Section 2 for a definition). In other words,

$$\mathbf{a}(n) \simeq D_1^{(n)} cr_n T_{\mathbf{a}}(S, \dots, S),$$

where $D_1^{(n)}$ indicates that we have applied D_1 successively with respect to each of the n variables, and S is the sphere spectrum, which is the unit of our symmetric monoidal category \mathcal{M}_S .

It turns out that this construction produces an operad even if we replace the triple $T_{\mathbf{a}}$ by a triple which is not necessarily associated with an operad. More precisely, we show that for every triple T , we can define an operadic multiplication on the sequence of objects $\{\mathbf{a}_T(n)\}$ given by $\mathbf{a}_T(n) \stackrel{def}{=} D_1^{(n)} cr_n T(S, \dots, S)$. We

refer to this operad as the operad induced by the triple T . Of course, it is not necessarily the case that the triple $T_{\mathbf{a}_T}$ associated with the operad \mathbf{a}_T is equivalent to T . For all X , $T(X)$ is always a T -algebra, and, when T and $T_{\mathbf{a}_T}$ are different, $T(X)$ may in addition have a structure of an \mathbf{a}_T -algebra, thus equipping $T(X)$ with two *different* algebraic structures - arising from the triple T and the operad \mathbf{a}_T .

Exploration of these structures is at the root of understanding the splitting of the Goodwillie tower of the triple T , as we prove the following theorem.

Theorem 1.4. *The Goodwillie tower of a triple T in \mathcal{M}_A splits at X if and only if $T(X)$ is an \mathbf{a}_T -ring spectrum, and the two algebra structures are compatible in some natural sense.*

The three splitting results described above form the core of this work.

The paper is structured as follows. In Section 2, we briefly recall some basic definitions from Goodwillie Calculus, as well as set up the notation. Theorem 1.2 is proved in Section 3. Section 4 is devoted to adopting P.May's two sided bar construction to the category $\mathcal{C}_{\mathbf{a}}$ of \mathbf{a} -algebras. In addition, geometric realizations and closed model structures on $\mathcal{C}_{\mathbf{a}}$ are discussed. In Section 5, we introduce the forgetful functor $U_{\mathbf{a}} : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$ and compute its layers D_n in terms of D_1 . Also, for a special class of operads (to which we refer as 'primitively generated operads'), we give an algebraic description of the Taylor tower of $U_{\mathbf{a}}$, which does not assume familiarity with Goodwillie Calculus. We employ these computations to prove Theorem 1.3 (as well as its 'Calculus free' analogue for the special class of primitively generated operads) in Section 6. There, we also discuss how the classical theorems of Leray and Hochschild, Kostant and Rosenberg can be recovered from our results. In Section 7, we show how a triple T induces an operad and discuss some examples. Section 8 explores the two algebra structures on $T(X)$ (produced by the triple itself and the induced operad), and uses these to derive necessary and sufficient conditions for T to be generalized homogeneous. Finally, in Section 9, we prove the technical results presented (without proof) in Section 5.

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2 Derivatives and Differentials

Let A be a cofibrant commutative S -algebra, and let \mathcal{M}_A be the category of A -modules. In this section we give a brief summary of relevant (for our purposes) constructions of Goodwillie Calculus for the category of functors from a pointed category \mathcal{C} to \mathcal{M}_A . It is no more than a restatement of the appropriate constructions from [14], which in turn is an application of [11], [12] and [13].

We begin by recalling the definition of the cross effects of a functor $F : \mathcal{C} \rightarrow \mathcal{A}$, where \mathcal{C} is a basepointed

category with finite coproducts and \mathcal{A} is an Abelian category. We denote the basepoint of \mathcal{C} by 0. The notion of cross effects dates back to S.Eilenberg and S.MacLane ([6]).

Definition 2.1. The n -th cross effect of F is the functor $cr_n F : \mathcal{C}^{\times n} \rightarrow \mathcal{A}$ defined inductively by

$$cr_1 F(M) \oplus F(*) \cong F(M)$$

$$cr_2 F(M_1, M_2) \oplus cr_1 F(M_1) \oplus cr_1 F(M_2) \cong cr_1 F(M_1 \vee M_2)$$

and in general,

$$cr_n F(M_1, \dots, M_n) \oplus cr_{n-1} F(M_1, M_3 \cdots M_n) \oplus cr_{n-1} F(M_2, M_3 \cdots M_n)$$

is equivalent to

$$cr_{n-1} F(M_1 \vee M_2, M_3 \cdots M_n).$$

To ease the notation we will denote the n -multifunctor $cr_n F(M, \dots, M)$ by $cr_n F(M)$.

Definition 2.2. Given a functor F from \mathcal{C} to \mathcal{M}_A , we say that F is degree n if $cr_{n+1} F$ is acyclic as a functor from $\mathcal{C}^{\times n+1}$ to \mathcal{M}_A . That is, $cr_{n+1} F$ is contractible when evaluated on a collection of $n+1$ objects in \mathcal{C} .

Denote the category of functors of $n+1$ variables from \mathcal{C} to \mathcal{A} that are reduced in each variable by $Func_*(\mathcal{C}^{\times n+1}, \mathcal{A})$. Let Δ^* be the functor from $Func_*(\mathcal{C}^{\times n+1}, \mathcal{A})$ to $Func_*(\mathcal{C}, \mathcal{A})$ obtained by composing a functor with the diagonal functor from \mathcal{C} to $\mathcal{C}^{\times n+1}$. When \mathcal{A} is the category of chain complexes over a commutative ring, the $(n+1)$ st cross effect is the right adjoint to Δ^* . Consequently, the functor $\perp_{n+1} \stackrel{def}{=} \Delta^* \circ cr_{n+1}$ is a cotriple on $Func_*(\mathcal{C}, \mathcal{A})$. For the case $\mathcal{A} = \mathcal{M}_A$, Δ^* and cr_{n+1} are not adjoint, however the composite functor $\perp_{n+1} = \Delta^* \circ cr_{n+1}$ is still a cotriple on $Func_*(\mathcal{C}, \mathcal{M}_A)$ (see Appendix of [17] for a proof).

Definition 2.3. Let $\perp_{n+1} = \Delta^* \circ cr_{n+1}$ be the above cotriple on $Func_*(\mathcal{C}, \mathcal{M}_A)$. For each $F \in Func_*(\mathcal{C}, \mathcal{M}_A)$, denote by $\perp_{n+1}^{*+1} F$ the simplicial object whose simplices are $\perp_{n+1}^{*+1} F$. We define P_n to be the functor from $Func_*(\mathcal{C}, \mathcal{M}_A)$ to $Func_*(\mathcal{C}, \mathcal{M}_A)$ given by

$$P_n F(X) = hoco\,fiber[|\perp_{n+1}^{*+1} F(X)| \xrightarrow{\epsilon} id(F(X))]$$

where $|\perp_{n+1}^{*+1} F(X)|$ is the geometric realization of the simplicial object. Furthermore, let $p_n : id \rightarrow P_n$ be the natural transformation obtained from the homotopy cofiber.

We note that the functor $P_n F(X)$ is degree n . In addition, if F is already of degree n , then $p_n : F \rightarrow P_n$ is an equivalence.

Next we produce a natural transformation $q_n : P_n \rightarrow P_{n-1}$.

Observe that we have the following formula relating the n 'th and $n+1$ 'st cross effects:

$$cr_{n+1} F(X_1, \dots, X_{n+1}) = cr_2(cr_n F(X_1, \dots, X_{n-1}, -))(X_n, X_{n+1}). \quad (1)$$

Fix an object X of \mathcal{C} . Let $G(Y) = cr_n F(X, \dots, X, Y)$. Then using the fold map, we have $cr_2 G(X, X) \rightarrow G(X \vee X) \rightarrow G(X)$, which in turn gives us

$$cr_2(cr_n F(X, \dots, X, -))(X, X) \rightarrow cr_n F(X, \dots, X, X).$$

Combine this map with the Equation 1 to produce a map $cr_{n+1} F(X, \dots, X) \rightarrow cr_n F(X, \dots, X)$, which induces the desired map q_n .

Definition 2.4. The n 'th layer or the n 'th derivative of F is the functor

$$D_n F(-)(X) \stackrel{def}{=} \text{hofiber}(q_n)(X).$$

In particular, the functor $D_1 F$ (which, by definition, is equivalent to $P_1 F$) will be referred to as “the derivative” or “the linearization” of F .

The linearization functor D_1 plays a rather central role, since the higher derivatives can be expressed in terms of D_1 . So following [14] we introduce the following notation. For a functor F of n variables, we denote by $D_1^i F(X_1, \dots, X_n)$ the derivative of F obtained by holding all but i 'th variable constant, in other words D_1 is applied to the single variable functor obtained by fixing all but i 'th variables. In addition, denote by $D_1^{(n)} F$ the multilinearized functor $D_1^n \dots D_1^2 D_1 F$.

The following proposition, which provides a description of objects in Goodwillie towers in terms of D_1 , is the combination of Proposition 3.9 and a special case of Proposition 3.1 of [14].

Proposition 2.5. *Let $F : \mathcal{C} \rightarrow \mathcal{M}_A$ be a functor into the category of A -modules. Then*

1. $D_n F$ is naturally equivalent to $D_1^{(n)} cr_n F_{h\Sigma_n}$,
2. $P_n cr_n F(X)$ is naturally equivalent to $D_1^{(n)} cr_n F(X)$.

We conclude this section with the definition of a *differential* of a functor, which mimics the notion of the directional derivative of a differentiable multi-variable function. See Section 5 of [14] for details.

Definition 2.6. Let $F : \mathcal{C} \rightarrow \mathcal{M}_A$ be a functor, and X and Y be objects in \mathcal{C} . The differential of F is the bifunctor defined by

$$\nabla F(X; Y) = D_1^X [\text{Fiber}(F(Y \vee X) \rightarrow F(Y))],$$

where the superscript in D_1^X indicates that the derivative is taken with respect to the variable X . Consequently, $\nabla F(X; Y)$ is linear in X but not necessarily in Y .

As a simple example, observe that the differential at $Y = 0$ is

$$\nabla F(X; 0) = D_1^X [\text{Fiber}(F(0 \vee X) \rightarrow F(0))] \cong D_1^X cr_1 F(X) = D_1 F(X).$$

3 Splitting of Goodwillie Towers

For a functor F into the category \mathcal{M}_A of A -modules, the data of the previous Section 2 assembles into a diagram (commonly referred to as the Goodwillie tower of F at X)

$$\begin{array}{ccccc}
 & & \dots & & \\
 & & \downarrow q_{n+1} & & \\
 F(X) & \xrightarrow{p_n} & P_n F(X) & \xleftarrow{d_n} & D_n F(X) \\
 & \searrow p_{n-1} & \downarrow q_n & & \\
 & & P_{n-1} F(X) & \xleftarrow{d_{n-1}} & D_{n-1} F(X) \\
 & & \downarrow & & \\
 & & \dots & &
 \end{array}$$

such that the homotopy limit $P_\infty F(X)$ of the inverse limit system $\{P_n F(X)\}$ is equivalent to $F(X)$ under suitable conditions. While the convergence issue will get addressed at various points throughout this work, here we concentrate on the question of developing conditions under which the above tower splits.

In this section alone, we will assume that the above tower does converge to $F(X)$. We do so to ease the notation while discussing the splitting problems. Naturally, once the tower is constructed, it will split or not split regardless of the object to which it converges. In fact, in what follows we could simply replace $F(X)$ by $P_\infty F(X)$ to eliminate the issue of convergence.

We begin by making a few simple observations that will lead to necessary conditions for splitting of the tower and will also simplify our terminology and arguments at later stages. Note that if the tower splits at X (i.e. $F(X) \simeq \Pi D_n F(X)$), the derivative map $F(X) \rightarrow D_1 F(X)$ has a section. More generally, by Proposition 2.5 we have that

$$D_n F(X) \simeq D_1^{(n)} cr_n F(X)_{h\Sigma_n} \cong D_1^{(n)} F(\vee_n X)_{h\Sigma_n}. \quad (2)$$

To see the second equivalence, recall that

$$cr_n F(M_1, \dots, M_n) \vee cr_{n-1} F(M_1, M_3, \dots, M_n) \vee cr_{n-1} F(M_2, M_3, \dots, M_n)$$

is isomorphic to $cr_{n-1} F(M_1 \vee M_2, M_3, \dots, M_n)$. However note that the terms $cr_{n-1} F(M_1, M_3, \dots, M_n)$ and $cr_{n-1} F(M_2, M_3, \dots, M_n)$ are constant in variables M_2 and M_1 respectively. Hence they vanish after we apply D_1^2 and D_1^1 . Consequently, $D_1^{(n)} cr_n F(M_1, \dots, M_n) \simeq D_1^{(n)} cr_{n-1} F(M_1 \vee M_2, M_3, \dots, M_n)$. Iterating this construction, we get the second part of Equation 2. Further observe that

$$D_1^1 F(\vee_n X) = D_1^1 F(X \vee \vee_{n-1} X) \simeq D_1^1 [F(X \vee \vee_{n-1} X) - F(\vee_{n-1} X)] = \nabla F(X; \vee_{n-1} X),$$

where by $F(X \vee \vee_{n-1} X) - F(\vee_{n-1} X)$ we mean the fiber of the obvious map $F(X \vee \vee_{n-1} X) \rightarrow F(\vee_{n-1} X)$ which has a section. We will use this notation in the future. (See Definition 2.6.) In particular, the question of the existence of a splitting of the derivative map $F(X) \rightarrow D_1 F(X)$ is equivalent to the existence of a splitting of the map $F(X) \rightarrow \nabla F(X; 0)$.

Our immediate objective is to show that, more generally, if the Goodwillie tower of F splits at $\vee_k X$ for all $k \geq 1$, then the natural maps

$$F(\vee_k X) = F(X \vee \vee_{k-1} X) \rightarrow \nabla F(X; \vee_{k-1} X) \quad (3)$$

also split. In other words, assuming that the Goodwillie tower of F at $\vee_k X$ decomposes into a product, we need to construct splittings

$$\nabla F(X; \vee_{k-1} X) \simeq \nabla \Pi D_n F(X; \vee_{k-1} X) \rightarrow \Pi D_n F(\vee_k X) \simeq F(\vee_k X).$$

To do so we analyze the individual factors on the two sides. Observe that

$$\begin{aligned} D_n F(\vee_k X) &\simeq D_1^{(n)} cr_n F(\vee_k X)_{h\Sigma_n} = D_1^{(n)} cr_n F(\vee_k X, \vee_k X, \dots, \vee_k X)_{h\Sigma_n} \\ &\simeq [D_1^{(n)} cr_n F(S, \vee_k X, \dots, \vee_k X) \wedge (\vee_k X)]_{h\Sigma_n} \simeq \dots \simeq D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (\vee_k X)^{\wedge n}, \end{aligned} \quad (4)$$

where S is the sphere spectrum, and the last n equivalences are by linearity of the derivative D_1 . See [19] for more on this. On the other hand,

$$\begin{aligned} \nabla D_n F(X; \vee_{k-1} X) &= D_1^1 [D_n F(X \vee \vee_{k-1} X) - D_n F(\vee_{k-1} X)] \\ &\simeq D_1^1 [D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (\vee_k X)^{\wedge n} - D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (\vee_{k-1} X)^{\wedge n}] \\ &\simeq D_1^1 [D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} ((\vee_k X)^{\wedge n} - (\vee_{k-1} X)^{\wedge n})] \\ &\simeq D_1^1 [D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (nX \wedge (\vee_{k-1} X)^{\wedge n-1} \vee \frac{n(n-1)}{2} X^{\wedge 2} \wedge (\vee_{k-1} X)^{\wedge n-2} \vee \dots \vee X^{\wedge n})] \\ &\simeq D_1^1 [D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (nX \wedge (\vee_{k-1} X)^{\wedge n-1})]. \end{aligned}$$

To obtain the last equivalence we made use of the fact that for $l \geq 2$, $D_1^1(X^{\wedge l} \wedge (\vee_{k-1} X)^{\wedge n-l})$ is contractible by Proposition 3.1 of [14] since $X^{\wedge l} \wedge (\vee_{k-1} X)^{\wedge n-l}$ is l -multireduced in the variable with respect to which the derivative is taken. Moreover, note that

$$D_1^1 [D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} (nX \wedge (\vee_{k-1} X)^{\wedge n-1})] \simeq D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} [\vee_n X \wedge (\vee_{k-1} X)^{\wedge n-1}],$$

to conclude that $\nabla D_n F(X; \vee_{k-1} X) \simeq D_1^{(n)} cr_n F(S) \wedge_{h\Sigma_n} [\vee_n X \wedge (\vee_{k-1} X)^{\wedge n-1}]$. We can restate the Map 3 in terms of this description and the description provided by Equation 4. It is simply the map induced by projections

$$(\vee_k X)^{\wedge n} \cong (X \vee \vee_{k-1} X)^{\wedge n} \cong X^{\wedge n} \vee (\vee_n X \wedge (\vee_{k-1} X)^{\wedge n-1}) \vee \dots \rightarrow \vee_n X \wedge (\vee_{k-1} X)^{\wedge n-1}.$$

Thus the desired splitting

$$\nabla D_n F(X; \vee_{k-1} X) \rightarrow D_n F(\vee_k X)$$

is produced by inclusions $\vee_n X \wedge (\vee_{k-1} X)^{\wedge n-1} \rightarrow (\vee_k X)^{\wedge n}$. Further, observe that the projections $\Pi D_n F(Y) \rightarrow D_n F(Y)$ give rise to morphisms $\nabla \Pi D_n F(X; \vee_{k-1} X) \rightarrow \nabla D_n F(X; \vee_{k-1} X)$, which in turn induce a map into the product

$$\nabla \Pi D_n F(X; \vee_{k-1} X) \rightarrow \Pi \nabla D_n F(X; \vee_{k-1} X),$$

producing a composite

$$\nabla F(X; \vee_{k-1} X) \simeq \nabla \Pi D_n F(X; \vee_{k-1} X) \rightarrow \Pi \nabla D_n F(X; \vee_{k-1} X) \rightarrow \Pi D_n F(\vee_k X) \simeq F(\vee_k X),$$

which is the desired splitting.

In fact, the existence of these splittings is also a sufficient condition for the towers at $\vee_k X$ for all $k \geq 1$ to split. More precisely, we prove the following theorem.

Theorem 3.1. *Let $F : \mathcal{C} \rightarrow \mathcal{M}_A$ be a functor from a pointed category \mathcal{C} to the category of A -modules, and let X be an object in \mathcal{C} . Then the following two statements are equivalent.*

1. *F and X are such that for all n , in addition to the natural derivative maps $F(\vee_n X) \rightarrow \nabla F(X; \vee_{n-1} X) \cong D_1^i F(\vee_n X)$ (for $1 \leq i \leq n$), there are n morphisms (one for each copy of X in $\vee_n X$) $\nabla F(X; \vee_{n-1} X) \rightarrow F(\vee_n X)$, such that the n composites*

$$\nabla F(X; \vee_{n-1} X) \rightarrow F(\vee_n X) \rightarrow \nabla F(X; \vee_{n-1} X) \tag{5}$$

are equivalences.

2. *The Goodwillie tower of F at $\vee_n X$ splits for all $n \geq 1$.*

Proof. Given our discussion preceding the theorem, we only need to show that the second statement follows from the first one.

First we show that under conditions of Statement **1**, the Goodwillie tower of F at X splits.

To begin, observe that for $n = 1$ the Equation 5 simply states that the derivative map $F(X) \rightarrow D_1 F(X)$ has a section $s : D_1 F(X) \rightarrow F(X)$, since $\nabla F(X; 0) \cong D_1 F(X)$. Composing the section s with $p_2 : F(X) \rightarrow P_2 F(X)$, we get a morphism $D_1 F(X) \rightarrow P_2 F(X)$.

More generally, we would like to produce similar maps for higher degrees, in other words, we are looking to construct sections $D_n F(X) \rightarrow P_{n+1} F(X)$ to q_n for all $n > 1$. To do so, consider the map

$$D_1^1 F(\vee_n X) \simeq \nabla F(X; \vee_{n-1} X) \rightarrow F(\vee_n X)$$

and apply D_1^2 to the two sides to get $D_1^2 D_1^1 F(\vee_n X) \rightarrow D_1^2 F(\vee_n X)$. Recalling that by assumption we have a map $D_1^2 F(\vee_n X) \cong \nabla F(X; \vee_{n-1} X) \rightarrow F(\vee_n X)$, we produce a composite $D_1^2 D_1^1 F(\vee_n X) \rightarrow F(\vee_n X)$. Iterating this construction, i.e. applying D_1^3, \dots, D_1^n in succession, and composing the resulting morphism with the ‘+’-map, we get

$$D_1^{(n)} F(\vee_n X) \rightarrow F(\vee_n X) \xrightarrow{F(+)} F(X). \quad (6)$$

Since the ‘+’-map is Σ_n -equivariant, the Equation 6 induces a morphism

$$D_n F(X) \simeq D_1^{(n)} cr_n F(X)_{h\Sigma_n} \xrightarrow{\simeq} D_1^{(n)} F(\vee_n X)_{h\Sigma_n} \rightarrow F(X) \rightarrow P_{n+1} F(X), \quad (7)$$

where the first equivalence $D_n F(X) \simeq D_1^{(n)} cr_n F(X)_{h\Sigma_n}$ is by Proposition 2.5. We induct on n to show that these maps produce the desired splittings.

For $n = 2$, recall that the morphism $D_1 F(X) \rightarrow F(X) \rightarrow P_2 F(X)$ is a splitting to $P_2 F(X) \xrightarrow{q_2} D_1 F(X)$, hence $P_2 F(X)$ is equivalent to $D_1 F(X) \vee D_2 F(X)$.

Now suppose for all $n \leq k$ we have that $P_n F(X)$ is equivalent to the coproduct of layers $D_1 F(X) \vee \dots \vee D_n F(X)$. Consider the diagram

$$\begin{array}{ccccc} D_{k+1} F(X) \simeq D_1^{(k+1)} cr_{k+1} F(X)_{h\Sigma_{k+1}} & \xrightarrow{d_{k+1}} & P_{k+1} F(X) & \xrightarrow{q_{k+1}} & P_k F(X) \\ & \nearrow & \downarrow q_{k+1} & & \downarrow q_k \\ D_k F(X) \simeq D_1^{(k)} cr_k F(X)_{h\Sigma_k} & \xrightarrow{d_k} & P_k F(X) & \xrightarrow{q_k} & P_{k-1} F(X) \end{array}$$

To see that $P_{k+1} F(X)$ splits we simply need to show that the composite

$$P_k F(X) \simeq P_{k-1} F(X) \vee D_1^{(k)} cr_k F(X)_{h\Sigma_k} \rightarrow F(X) \xrightarrow{p_{k+1}} P_{k+1} F(X) \xrightarrow{q_{k+1}} P_k F(X) \quad (8)$$

is equivalent to the identity map. Here the first map on the component $P_{k-1} F(X)$ exists because by inductive hypothesis, $P_{k-1} F(X)$ is equivalent to the coproduct of layers $D_1 \vee \dots \vee D_{k-1}$, and the maps on layers are defined via Equation 7. It is enough to prove that Map 8 is equivalent to the identity map on the component $D_1^{(k)} cr_k F(X)_{h\Sigma_k}$, because the identity on the component $P_{k-1} F(X)$ follows by inductive hypothesis. In other words, we need show that the composite

$$D_1^{(k)} cr_k F(X)_{h\Sigma_k} \rightarrow F(X) \xrightarrow{p_k} P_k F(X), \quad (9)$$

which is the restriction of Map 8 to $D_1^{(k)} cr_k F(X)_{h\Sigma_k}$, is equivalent to the map $D_1^{(k)} cr_k F(X)_{h\Sigma_k} \xrightarrow{d_k} P_k F(X)$ of the Goodwillie tower of $F(X)$. Consider the commutative diagram

$$\begin{array}{ccc} cr_k F(X) & \xrightarrow{f} & F(X) \\ s \left(\begin{array}{c} \downarrow p_k \\ \downarrow p_k \end{array} \right. & & \downarrow p_k \\ P_k cr_k F(X) & \xrightarrow{P_k(f)} & P_k F(X) \end{array} \quad (10)$$

where f is the ‘+’-map $cr_k F(X) \rightarrow F(X)$, two vertical maps pointing down are from Goodwillie towers of $F(X)$ and $cr_k F(X)$, and the upward vertical map s exists by Statement 1 of the Theorem since by Proposition 2.5, $P_k cr_k F(X)$ is equivalent to $D_1^{(k)} cr_k F(X)$. Consequently, s is simply a successive application of sections to the derivative map as in Equation 6, and thus, is itself a section to p_k .

Observe that the composite $p_k \circ f \circ s : D_1^{(k)} cr_k F(X) \rightarrow P_k F(X)$ factors through the homotopy orbits to produce the Map 9, while the map $P_k(f)$ induces $d_k : D_1^{(k)} cr_k F(X)_{h\Sigma_k} \rightarrow P_k F(X)$. Hence, the commutativity of Diagram 10 implies the desired equivalence of the Map 9 and the map d_k .

Thus, we are allowed to conclude that the map $P_k \rightarrow P_{k+1}$ given in Equation 8 is a splitting to q_{k+1} , proving that the Goodwillie tower of F at X splits.

Now let $Y = \vee_t X$ for some $t > 1$. By what we just proved, to show that the Goodwillie tower of F splits at Y , it is enough to produce splittings for derivative maps $F(\vee_n Y) \rightarrow D_1^i F(\vee_n Y)$. We will only consider the case $i = 1$ and refer to the symmetry of arguments for all other i 's.

Recall that $D_1^1 F(\vee_n Y) = D_1^1 F(Y \vee \vee_{n-1} Y)$ is linear in the first variable. Hence,

$$D_1^1 F(Y \vee \vee_{n-1} Y) \simeq D_1^1 F(X_1 \vee \vee_{n-1} Y) \vee \cdots \vee D_1^1 F(X_t \vee \vee_{n-1} Y),$$

where $X_1 = \cdots = X_t = X$. The indexing is introduced to help keep track of different summands. By Statement 1, we have a morphism

$$D_1^1 F(X \vee \vee_{n-1} Y) = D_1^1 F(X \vee \vee_{t(n-1)} X) \rightarrow F(X \vee \vee_{t(n-1)} X),$$

which is a splitting to the derivative map $F(X \vee \vee_{t(n-1)} X) \rightarrow D_1^1 F(X \vee \vee_{t(n-1)} X)$. Thus, we get a morphism

$$\begin{aligned} D_1^1 F(Y \vee \vee_{n-1} Y) &\simeq \bigvee_{i=1}^t D_1^1 F(X_i \vee \vee_{n-1} Y) \rightarrow \bigvee_{i=1}^t F(X_i \vee \vee_{n-1} Y) = \bigvee_{i=1}^t F(X_i \vee \vee_{t(n-1)} X) \\ &\rightarrow F\left(\bigvee_{i=1}^t X_i \vee \vee_{t(n-1)} X\right) = F(Y \vee \vee_{n-1} Y), \end{aligned}$$

where the last map is induced by obvious inclusions $X_i \vee \vee_{t(n-1)} X \rightarrow \vee_{i=1}^t X_i \vee \vee_{t(n-1)} X$. To see that this map is a section to the derivative $F(Y \vee \vee_{n-1} Y) \rightarrow D_1^1 F(Y \vee \vee_{n-1} Y)$, we need to show that for each i , the composite

$$\begin{aligned} D_1^1 F(X_i \vee \vee_{n-1} Y) &\rightarrow F(X_i \vee \vee_{n-1} Y) \rightarrow F\left(\bigvee_{i=1}^t X_i \vee \vee_{n-1} Y\right) = F(Y \vee \vee_{n-1} Y) \\ &\rightarrow D_1^1 F(Y \vee \vee_{n-1} Y) \rightarrow D_1^1 F(X_i \vee \vee_{n-1} Y) \end{aligned} \tag{11}$$

is equivalent to the identity. Observe that the last two maps in the above composite fit into the following

commutative diagram

$$\begin{array}{ccc}
F(Y \vee \vee_{n-1} Y) & \longrightarrow & D_1^1 F(Y \vee \vee_{n-1} Y) \\
\downarrow & & \downarrow \\
F(X_i \vee \vee_{n-1} Y) & \longrightarrow & D_1^1 F(X_i \vee \vee_{n-1} Y)
\end{array}$$

where the vertical maps are induced by projections $Y = \bigvee_{i=1}^t X_i \rightarrow X_i$. Consequently, we can rewrite the Composite 11 as

$$\begin{aligned}
D_1^1 F(X_i \vee \vee_{n-1} Y) &\rightarrow F(X_i \vee \vee_{n-1} Y) \rightarrow F\left(\bigvee_{i=1}^t X_i \vee \vee_{n-1} Y\right) \\
&\rightarrow F(X_i \vee \vee_{n-1} Y) \rightarrow D_1^1 F(X_i \vee \vee_{n-1} Y),
\end{aligned}$$

which is the identity since the first map $D_1^1 F(X_i \vee \vee_{n-1} Y) \rightarrow F(X_i \vee \vee_{n-1} Y)$ is a section to the derivative map. \square

4 Algebras over Operads and Bar Construction

We begin this section by recalling the definition of operads. Operads can be defined in any symmetric monoidal category. However, since our interests lie primarily in the category of S -modules (or more generally in the category of A -modules), we will focus our discussion around these categories, though the category of chain complexes $Ch(K)$ over a commutative ring K will often get utilized as well to produce examples. Recall that the symmetric monoidal operation in the category of S -modules is the smash product \wedge .

Definition 4.1. [See [15]] An operad is a sequence of objects $\mathbf{a}(k)$, $k \geq 0$, carrying an action of symmetric groups Σ_k , with products

$$\gamma : \mathbf{a}(k) \wedge \mathbf{a}(j_1) \wedge \cdots \wedge \mathbf{a}(j_k) \rightarrow \mathbf{a}(j_1 + \cdots + j_k)$$

which are unital, equivariant and associative in the following sense.

(a) The following associativity diagrams commute, where $\Sigma_{j_s} = j$ and $\Sigma_{i_t} = i$; also set $g_s = j_1 + \cdots + j_s$ and $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$ for $1 \leq s \leq k$:

$$\begin{array}{ccc}
\mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_s)\right) \wedge \left(\bigwedge_{r=1}^j \mathbf{a}(i_r)\right) & \xrightarrow{\gamma \wedge Id} & \mathbf{a}(j) \wedge \left(\bigwedge_{r=1}^j \mathbf{a}(i_r)\right) \\
\downarrow \text{shuffle} & & \downarrow \gamma \\
\mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_s)\right) \wedge \left(\bigwedge_{q=1}^{j_s} \mathbf{a}(i_{g_{s-1}+q})\right) & \xrightarrow{Id \wedge (\wedge_s \gamma)} & \mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(h_s)\right) \\
& & \uparrow \gamma
\end{array}$$

(b) The following unit diagrams commute:

$$\begin{array}{ccc}
\mathbf{a}(k) \wedge S^{\wedge k} & \xrightarrow{\cong} & \mathbf{a}(k) \\
\text{Id} \wedge \eta^k \downarrow & \nearrow \gamma & \\
\mathbf{a}(k) \wedge \mathbf{a}(1)^{\wedge k} & &
\end{array}
\qquad
\begin{array}{ccc}
S \wedge \mathbf{a}(j) & \xrightarrow{\cong} & \mathbf{a}(j) \\
\eta \wedge \text{Id} \downarrow & \nearrow \gamma & \\
\mathbf{a}(1) \wedge \mathbf{a}(j) & &
\end{array}$$

(c) The following equivariance diagrams commute, where $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$, the permutation $\sigma(j_1, \dots, j_k) \in \Sigma_j$ permutes k blocks of letters as σ permutes k letters, and $\tau_1 \oplus \dots \oplus \tau_k \in \Sigma_k$ is the block sum:

$$\begin{array}{ccc}
\mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_s) \right) & \xrightarrow{\sigma \wedge \sigma^{-1}} & \mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_{\sigma_s}) \right) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathbf{a}(j) & \xrightarrow{\sigma(j_{\sigma(1)}, \dots, j_{\sigma(k)})} & \mathbf{a}(j)
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_s) \right) & \xrightarrow{\text{Id} \wedge \bigwedge_1^k \tau_i} & \mathbf{a}(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}(j_s) \right) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathbf{a}(j) & \xrightarrow{\tau_1 \oplus \dots \oplus \tau_k} & \mathbf{a}(j)
\end{array}$$

Our main interest is in the categories of algebras over operads. For detailed good discussions on general theory of algebras over operads we refer to [15] and [8]. Here we recall one of the equivalent definitions with a slight modification. In what follows, we are mostly going to be concerned with non-unital algebras. This corresponds to the additional hypothesis of $\mathbf{a}(0) \cong 0$ in the above definition, which will be assumed throughout this paper unless specified otherwise.

We denote the triple associated to \mathbf{a} by $T_{\mathbf{a}}$:

$$T_{\mathbf{a}}(X) = \bigoplus_{n=0}^{\infty} \mathbf{a}(n) \wedge_{\Sigma_n} X^{\wedge n}.$$

Of course, as pointed out, for us the 0-th summand is redundant. Let $\mathcal{C}_{\mathbf{a}}$ be the category of algebras over the operad \mathbf{a} , which coincides with the category of algebras over the triple $T_{\mathbf{a}}$. In other words, if B is in $\mathcal{C}_{\mathbf{a}}$, it is equipped with a collection of maps

$$\mathbf{a}(n) \wedge_{\Sigma_n} B^{\wedge n} \rightarrow B,$$

which make the obvious coherence diagrams commute. We will refer to algebras over an operad \mathbf{a} as \mathbf{a} -algebras.

To get a better homotopical control over the category of algebras $\mathcal{C}_{\mathbf{a}}$, we consider only the operads \mathbf{a} which arise from operads in simplicial sets. Before making this statement more explicit we remark that this is not very restrictive, as nearly all operads in the category of spectra that appear in literature are induced by an operad in simplicial sets.

Observe that if $\mathbf{b} = \{\mathbf{b}(n)\}$ is an operad in the category of simplicial sets, then \mathbf{a} with $\mathbf{a}(n) = S \otimes \mathbf{b}(n)$ is an operad in \mathcal{M}_S . Here we employed the fact that the category of S -modules is a tensored category. To see the operadic multiplications, recall that there are natural isomorphisms

$$(S \otimes \mathbf{b}(k)) \wedge (S \otimes \mathbf{b}(j_1)) \wedge \dots \wedge (S \otimes \mathbf{b}(j_k)) \cong S \otimes \mathbf{b}(k) \otimes \mathbf{b}(j_1) \otimes \dots \otimes \mathbf{b}(j_k).$$

Hence we can use the multiplication of \mathbf{b} to produce the operadic multiplication on \mathbf{a} . This is what we mean when discussing operads in \mathcal{M}_S induced by operads in the category of simplicial sets.

From this point on, unless specifically said otherwise, the word ‘operad’ means a cofibrant operad in the category of S -modules that arises from simplicial sets.

The following Proposition is part of Proposition 1.6 of [10].

Proposition 4.2. *If we define a map $X \rightarrow Y$ of \mathbf{a} -algebras to be a weak equivalence or a fibration if it is a weak equivalence or a fibration of S -modules, then the category of \mathbf{a} -algebras $\mathcal{C}_{\mathbf{a}}$ becomes a cofibrantly generated simplicial model category. In particular, it is equipped with a cofibrant replacement functor.*

The proof of this proposition is sketched in Section 1 of [10], for more details see also [3].

The following is a variant of the two sided bar construction of P.May ([18]).

Denote by $\mu_{\mathbf{a}}$ and $\eta_{\mathbf{a}}$ the multiplication and the unit respectively of the cotriple $T_{\mathbf{a}}$. When no confusion can arise we will omit the subscript \mathbf{a} from the notation. For an \mathbf{a} -algebra (C, ξ) , let $B_*^{\mathbf{a}}C$ denote the simplicial object with n 'th term $B_n^{\mathbf{a}}C = T_{\mathbf{a}}^{n+1}C$ and face and degeneracy operators given by

$$\begin{aligned} d_i &= T_{\mathbf{a}}^i \mu_{T_{\mathbf{a}}^{n-i-1}}, \text{ for } 0 \leq i < n, \text{ and } d_n = T_{\mathbf{a}}^n \xi \\ s_i &= T_{\mathbf{a}}^{i+1} \eta_{T_{\mathbf{a}}^{n-i}}, \text{ for } 0 \leq i \leq n. \end{aligned}$$

This construction can be interpreted in the language of [18] if we consider $T_{\mathbf{a}} : \mathcal{M}_A \rightarrow \mathcal{C}_{\mathbf{a}}$ as the functor left adjoint to the forgetful functor $U : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$. Then $B_*^{\mathbf{a}}C$ is the same as the object $B_*(T_{\mathbf{a}}, UT_{\mathbf{a}}, UC)$ in notation of [18]. Consequently, we can consider $\xi : B_*^{\mathbf{a}}C \rightarrow C$ as an augmented simplicial A -module for which one defines a contraction using the unit map $\eta : C \rightarrow T_{\mathbf{a}}C$. Here we have omitted the forgetful functor U from our notations and will continue to do so, as it is evident from the context.

Of course for the bar construction $B_*^{\mathbf{a}}C$ to be computationally useful we need to be allowed to work with it level-wise. To make this more precise we recall the notion of properness. To avoid confusion, here (and in the future) we use the term ‘c-cofibration’ (for classical cofibration) to distinguish it from cofibrations that are part of the model category structure. In other words, a c-cofibration of A -modules is simply a map $i : M \rightarrow \overline{M}$ of A -modules that satisfies the homotopy extension property in the category of A -modules. Of course, all cofibrations are c-cofibrations, but not conversely.

Definition 4.3. Let K_* be a simplicial spectrum and let $sK_q \subset K_q$ be the ‘union’ of the subspectra $s_j K_{q-1}$, $0 \leq j < q$. A simplicial A -module K_* is proper if the canonical map of A -modules $sK_q \rightarrow K_q$ is a c-cofibration for each $q \geq 0$.

The main reason that proper simplicial A -modules K_* are computationally useful is that one is allowed to work with them level-wise. More precisely, one can use the simplicial filtration to construct a well-behaved

spectral sequence that converges to $\pi_*(E \wedge K)$ for any spectrum E (Theorem X.2.9 of [7]). In particular, if $f : K_* \rightarrow L_*$ is a map of proper simplicial A -modules which is a weak equivalence level-wise, then the geometric realization $|f|$ of f is also a weak equivalence.

Lemma 4.4. *For a cofibrant \mathbf{a} -algebra C , $B_*^{\mathbf{a}}C$ is a proper simplicial A -module.*

Proof. The condition of properness involves only the degeneracy operators (and not the face maps) of a simplicial A -module. As it is evident from the definition of the bar construction the degeneracies are constructed from the unit map $\eta : C \rightarrow T_{\mathbf{a}}C$. This map has an obvious section $T_{\mathbf{a}}C \rightarrow C$ (given by the multiplication map) which is a map of A -modules. Thus, it satisfies the homotopy extension property. Similarly one shows that all degeneracies are c-cofibrations. \square

Note that the notion of geometric realization of the simplicial \mathbf{a} -algebra $B_*^{\mathbf{a}}C$ is somewhat ambiguous, as one could realize this object in the category of A -modules, or alternatively, in the category of \mathbf{a} -algebras, i.e. internally. We denote the realization in the category of \mathbf{a} -algebras by $|-|_{\mathbf{a}}$. The following proposition shows that the two realizations are the same. The proof presented here is based on an argument suggested by Paul Goerss. Proposition VII.3.3 of [7] provides a different, categorical approach.

Proposition 4.5. *The geometric realization in the category of \mathbf{a} -algebras is isomorphic to the geometric realization in the category of A -modules.*

Proof. First observe that it is sufficient to prove that the natural map

$$|T_{\mathbf{a}}M_*| \rightarrow T_{\mathbf{a}}|M_*| \tag{12}$$

is an isomorphism, where M_* is a simplicial A -module. Indeed, if this is the case then for any simplicial \mathbf{a} -algebra C_* , the geometric realization as modules is an \mathbf{a} -algebra via the composite (multiplication) map

$$T_{\mathbf{a}}|C_*| \simeq |T_{\mathbf{a}}C_*| \rightarrow |C_*|.$$

Consequently, by an argument similar to that in the proof of Proposition 3.3 of [7], we get that for any simplicial \mathbf{a} -algebra D_* , there is an isomorphism

$$\mathcal{C}_{\mathbf{a}}(|C_*|, D_*) \cong s\mathcal{C}(C_*, F_{\mathbf{a}}(\Delta_*, D_*)),$$

where $F_{\mathbf{a}}$ is the internal function space functor. Recalling that $F_{\mathbf{a}}(\Delta_*, -)$ is right adjoint to the geometric realization functor $|-|_{\mathbf{a}}$, we conclude that, by Yoneda's lemma, $|C_*|_{\mathbf{a}} \simeq |C_*|$.

Thus, it remains to show that Morphism (12) is an equivalence. Note that as a left adjoint, geometric realization commutes with coproducts and the functor $\mathbf{a}(n) \wedge_{\Sigma_n} (-)$. Hence it is enough to show that $|M_*^{\wedge n}|$

is equivalent to $|M_*|^{\wedge n}$. Further recalling that geometric realization commutes with the spectrification functor, we reduce the problem to showing that for any pointed simplicial sets X_* and Y_* , the natural map $|X_* \wedge Y_*| \rightarrow |X_*| \wedge |Y_*|$ is an equivalence.

However, this is an immediate consequence of the Eilenberg-Zilber theorem, once we observe that both $|X_* \wedge Y_*|$ and $|X_*| \wedge |Y_*|$ can be obtained from the bisimplicial space

$$([m], [n]) \mapsto X_m \wedge Y_n,$$

with $|X_* \wedge Y_*|$ being the geometric realization of the diagonal, and $|X_*| \wedge |Y_*|$ resulting from realizing in the m -direction first and then in the n -direction. □

As an immediate consequence of this proposition, we have that $|B_*^{\mathbf{a}}C|$ and C are weakly equivalent as \mathbf{a} -algebras. Indeed, we already observed that $|B_*^{\mathbf{a}}C|$ and C are homotopy equivalent as A -modules (via the contraction to the augmentation map), hence the weak equivalence as algebras follows from the above proposition.

5 The Forgetful Functor $U_{\mathbf{a}} : C_{\mathbf{a}} \rightarrow \mathcal{M}_A$

In this section we construct a tower of functors that approximates the forgetful functor $U_{\mathbf{a}} : C_{\mathbf{a}} \rightarrow \mathcal{M}_A$ from the category of \mathbf{a} -algebras to the category of A -modules. As noted in the previous section, we are still assuming that our operads arise from simplicial sets.

Roughly speaking, we say that the functor $F(X)$ is approximated by the tower of functors $P_n F(X)$ if they assemble into a diagram

$$\begin{array}{ccc}
 & & \dots \\
 & & \downarrow q_{n+1} \\
 F(X) & \xrightarrow{p_n} & P_n F(X) \\
 & \searrow p_{n-1} & \downarrow q_n \\
 & & P_{n-1} F(X) \\
 & & \downarrow \\
 & & \dots
 \end{array}$$

where p_i 's and q_i 's are natural transformations, such that under suitable conditions the homotopy limit of $\{P_n F(X)\}$ is weakly equivalent to $F(X)$.

As noted, one of the objectives of this work is to determine when the given \mathbf{a} -algebra decomposes into a direct sum, or more precisely, what are sufficient conditions for the algebra to be free. The idea is to construct a tower of functors approximating the above forgetful functor, and search for conditions under which this tower splits. We will return to the relationship between freeness and splitting towers, as well as to the precise meaning of the term ‘approximate’ at a later point.

The motivation for the ‘approximating’ tower of functors comes from Goodwillie Calculus. In particular, a similar tower is constructed in [14] for the discrete commutative case and in [23] for the E_∞ case. However, as noted, the primary objective of this part of the paper is to develop splitting criteria for algebras over operads, and in a large number of cases we are able to obtain and give a nice description of such splittings without having to resort to the terminology of Goodwillie Calculus.

The downside is an additional condition on the operad. For algebras over a general operad, we are still able to prove a splitting result, but the pieces into which our algebras decompose are somewhat less transparent, and Calculus language could not be avoided. It should be noted though that these ‘two’ decompositions are essentially the same in their common range; we will comment more on this at a later stage.

On an organizational point, this and the section following it, are constructed in such a manner that the results involving Goodwillie Calculus are clearly marked and can be omitted by the reader (if he/she chooses to do so) without affecting the rest of these two sections.

We begin by defining a functor $Q_n : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$ for all $n > 1$ by the following pushout diagram in the category of A -modules

$$\begin{array}{ccc} \mathbf{a}(n) \wedge_{\Sigma_n} C^{\wedge n} & \longrightarrow & * \\ \downarrow \mu & & \downarrow \\ C & \longrightarrow & Q_n(C). \end{array}$$

This functor already has some of the ingredients we require, namely it is equipped with a natural transformation $U_{\mathbf{a}} \rightarrow Q_n$, which is evident from the definition of Q_n . Furthermore, under suitable connectivity assumptions on an \mathbf{a} -algebra C , the connectivity of the maps $U_{\mathbf{a}}(C) \rightarrow Q_n(C)$ increases with n . However, we want to work with $\mathcal{C}_{\mathbf{a}}$ not only up to isomorphism, but up to a weak equivalence. Thus, we desire functors which preserve weak equivalences. Usually this is achieved by considering the derived version of the functor, i.e. functors get evaluated not at the objects themselves but at their cofibrant replacements. In addition, we would like our functors to be computationally friendly, which is often accomplished by constructing a cofibrant replacement via a resolution by free objects.

To that end, we introduce the following definition. In what follows, ΓC is the cofibrant replacement

functor in the category of \mathbf{a} -algebras, and it exists by Proposition 4.2.

Definition 5.1. For an \mathbf{a} -algebra C , define $I/I^n(C)$ to be the A -module

$$I/I^n(C) = Q_n(\Gamma B_*^{\mathbf{a}}\Gamma C)$$

where Q_n is applied level-wise. We will also write

$$I^n/I^{n+1}(C) \stackrel{def}{=} \text{hofiber}[I/I^{n+1}(C) \rightarrow I/I^n(C)].$$

Remark 5.2. Note that even when C is cofibrant as an A -module, $B_*^{\mathbf{a}}C$ is not necessarily cofibrant. Of course, $T_{\mathbf{a}}$ takes cofibrant A -modules to cofibrant \mathbf{a} -algebras, however, cofibrant \mathbf{a} -algebras are not necessarily cofibrant as A -modules. Consequently, when we apply $T_{\mathbf{a}}$ again, the resulting \mathbf{a} -algebra may no longer be cofibrant. Hence, since simplices of $B_*^{\mathbf{a}}C$ are formed by multiple applications of $T_{\mathbf{a}}$, they may not be cofibrant. Thus, we are forced to replace $B_*^{\mathbf{a}}\Gamma C$ by $\Gamma B_*^{\mathbf{a}}\Gamma C$, where the outer Γ is applied level-wise. Naturally, in doing so, we lose some of the computational advantage. However, as is illustrated in the proof of the next proposition, this problem can be overcome.

The following Proposition 5.5 gives us a good description of the layers (fibers) of the approximating tower of functors. More precisely it allows us to express the higher layers in terms of the first one. As a condition on the operad is required, we introduce a definition first.

Definition 5.3. We say that the operad \mathbf{a} is primitively generated if the square

$$\begin{array}{ccc} \mathbf{a}(n) \wedge_{\Sigma_n} T_{\mathbf{a}}(X)^{\wedge n} & \longrightarrow & T_{\mathbf{a}}(X) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \bigoplus_{i \leq n-1} \mathbf{a}(i) \wedge_{\Sigma_i} X^{\wedge i} \end{array}$$

is Cartesian for all X and $n \geq 1$.

Note in particular, that for primitively generated operads, the multiplication map $\mathbf{a}(n) \wedge \mathbf{a}(1) \wedge \cdots \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(n)$ is an equivalence.

Remark 5.4. This is both a remark and an example as its purpose is to explain our choice of terminology in Definition 5.3, as well as to show that quadratic operads (see [9]) are primitively generated. To do so, we analyze the primitively generated operads in the category of differential graded complexes more closely. Let \mathbf{a} be such an operad. To understand the top horizontal map in the above diagram better, observe that

$$(T_{\mathbf{a}}X)^{\wedge n} = \bigoplus_{(j_1, \dots, j_n)} \mathbf{a}(j_1) \wedge \cdots \wedge \mathbf{a}(j_n) \wedge_{\Sigma_{j_1} \times \cdots \times \Sigma_{j_n}} C^{\wedge(j_1 + \cdots + j_n)}.$$

Consequently, for say $n = 2$, that map is given by components

$$\mathbf{a}(2) \wedge [\mathbf{a}(s) \wedge \mathbf{a}(t) \oplus \mathbf{a}(t) \wedge \mathbf{a}(s)] \wedge_{\Sigma_2 \ltimes (\Sigma_t \times \Sigma_s)} X^{\wedge(s+t)} \rightarrow \mathbf{a}(s+t) \wedge_{\Sigma_{s+t}} X^{\wedge(s+t)},$$

which are a combination of the ‘+’-map and the operadic multiplication. Since for all X all summands of degree n and higher must get annihilated in the above diagram, and for $n = 2$, the component $\mathbf{a}(k) \wedge_{\Sigma_k} X^{\wedge k}$ gets “hit” by

$$\bigoplus_{t+s=k} \mathbf{a}(2) \wedge [\mathbf{a}(s) \wedge \mathbf{a}(t) \oplus \mathbf{a}(t) \wedge \mathbf{a}(s)] \wedge_{\Sigma_2 \ltimes (\Sigma_t \times \Sigma_s)} X^{\wedge(s+t)},$$

with t and s greater than 0, then we can assert that $\mathbf{a}(k)$ is produced by the objects $\mathbf{a}(t)$ of degree lower than k . Inducting down on k , we conclude that all objects in the sequence of the operad \mathbf{a} are “produced” by $\mathbf{a}(1)$ and $\mathbf{a}(2)$. To make this more precise we recall the notion of a free operad on a symmetric sequence $\mathbf{v}(l)$. In other words, for each l , $\mathbf{v}(l)$ is an object in the symmetric monoidal category with an action of the symmetric group Σ_l . The forgetful functor from the category of operads to symmetric sequences has a left adjoint \mathbb{T} endowed with a natural map $\eta : \mathbf{v} \rightarrow \mathbb{T}\mathbf{v}$, which satisfies the following universal property. If \mathbf{p} is an operad, and $\bar{\eta} : \mathbf{v} \rightarrow \mathbf{p}$ is a map of symmetric sequences, then there is a unique morphism of operads $\phi : \mathbb{T}\mathbf{v} \rightarrow \mathbf{p}$, such that $\bar{\eta} = \phi\eta$. The operad $\mathbb{T}\mathbf{v}$ is referred to as the free operad on \mathbf{v} . A detailed description of the free operad functors in terms of trees is provided in [9].

In addition, it is possible to define the notion of an operad *ideal*, in such a way that $\mathbf{c} \subset \mathbf{a}$ is an operad ideal if and only if the operadic multiplication of \mathbf{a} induces an operadic multiplication on the quotient symmetric sequence \mathbf{a}/\mathbf{c} .

Thus, using this new language, we assert that the primitively generated operads are quotients of the free operad on symmetric sequences $\mathbf{v}(l)$ with $\mathbf{v}(l) = 0$ for all $l > 2$. Conversely, suppose the operad \mathbf{a} is such that the operadic multiplication $\mathbf{a}(2) \wedge \mathbf{a}(1) \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(2)$ is an equivalence and the generators of the operad are all in degrees 1 and 2. Then the top horizontal map

$$\mathbf{a}(n) \wedge_{\Sigma_n} T_{\mathbf{a}}(X)^{\wedge n} \rightarrow T_{\mathbf{a}}(X)$$

of the diagram in Definition 5.3 misses the components of degree less than n , and “hits” all the summands starting from n , thus making the diagram Cartesian.

Quadratic operads provide examples of these as they are defined to be the operads whose generators are concentrated in degree 2 while the relations are in degree 3; see [9]. Consequently, all the well known quadratic operads, such the *associative*, *commutative*, and *Lie* operads are examples of primitively generated operads.

Now we state the promised result.

Proposition 5.5. *Let \mathbf{a} be a primitively generated operad. Then for every \mathbf{a} -algebra C , there is a weak equivalence of simplicial objects*

$$I^n/I^{n+1}(C) \simeq \mathbf{a}(n) \wedge_{h\Sigma_n} \bigwedge^n I/I^2(C),$$

where $\bigwedge^n I/I^2$ is the n 'th smash power of I/I^2 .

This is a generalization of Proposition 2.4 of [23], where a similar result is proved for E_∞ operads. In proving this proposition we use some of the techniques developed in [23] and [1]. We will need a key technical lemma, which we will prove later, in Section 9, in order not to distract from our task at hand.

Lemma 5.6. *Let C be a cofibrant \mathbf{a} -algebra and $\gamma : Y \rightarrow T_{\mathbf{a}}^n C$ a cell A -module approximation.*

1. *Then the induced map*

$$\mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i} \rightarrow \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i}$$

is a weak equivalence for all $i > 0$.

2. *The projection $\mathbf{a}(i) \wedge_{h\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i} \rightarrow \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i}$ is an equivalence for all n and i .*

Proof. (of Proposition 5.5) As it was noted earlier, by applying the functor $B_*^{\mathbf{a}}$ we gained some computational advantages (as it is the analogue of taking a free resolution in discrete algebra), however some of it was lost when we were forced to take a cofibrant replacement. Hence, the first objective is to show that we get equivalent constructions even if we forgo taking cofibrant replacements, in other words, we begin by showing that $I/I^n(C)$ is weakly equivalent to $Q_n(B_*^{\mathbf{a}}\Gamma C)$ as modules.

Let $\gamma : Y \rightarrow T_{\mathbf{a}}^k C$ be a cell A -module approximation of $T_{\mathbf{a}}^k C$, where C is a cofibrant \mathbf{a} -algebra. Then by Lemma 5.6, we get that $T_{\mathbf{a}}\gamma : T_{\mathbf{a}}Y \rightarrow T_{\mathbf{a}}T_{\mathbf{a}}^k C$ is a weak equivalence, since it is a coproduct of weak equivalences. Hence so is $\Gamma T_{\mathbf{a}}Y \rightarrow \Gamma T_{\mathbf{a}}T_{\mathbf{a}}^k C$. In fact, since the two algebras involved are cofibrant, this last map is a simplicial homotopy equivalence. Consequently, the map $Q_n(\Gamma T_{\mathbf{a}}Y) \rightarrow Q_n(\Gamma T_{\mathbf{a}}T_{\mathbf{a}}^k C)$ is also a simplicial homotopy equivalence, as any functor preserves simplicial homotopy equivalences.

Now consider the following commutative diagram:

$$\begin{array}{ccc} Q_n(\Gamma T_{\mathbf{a}}Y) & \longrightarrow & Q_n(\Gamma T_{\mathbf{a}}T_{\mathbf{a}}^k C) \\ \downarrow & & \downarrow \\ Q_n(T_{\mathbf{a}}Y) & \longrightarrow & Q_n(T_{\mathbf{a}}T_{\mathbf{a}}^k C). \end{array}$$

As we just argued, the top horizontal map is a homotopy equivalence. Since $T_{\mathbf{a}}Y$ is a cofibrant \mathbf{a} -algebra, the cofibrant replacement weak equivalence $\Gamma T_{\mathbf{a}}Y \rightarrow T_{\mathbf{a}}Y$ is in fact a homotopy equivalence. Hence the left vertical map is also a homotopy equivalence. To see that the bottom horizontal map is a weak equivalence,

we analyze $Q_n(T_{\mathbf{a}}Y)$ more closely. By definition, it is the cofiber of the map

$$\mathbf{a}(n) \wedge_{\Sigma_n} \left[\bigoplus_{i=1}^{\infty} \mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i} \right]^{\wedge n} \rightarrow \bigoplus_{i=1}^{\infty} \mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i},$$

which, since \mathbf{a} is primitively generated, is equivalent to $\bigoplus_{i \leq n-1} \mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i}$. Similarly,

$$Q_n(T_{\mathbf{a}}T_{\mathbf{a}}^k C) \simeq \bigoplus_{i \leq n-1} \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}}^k C)^{\wedge i}.$$

The equivalence of the bottom horizontal arrow follows from Lemma 5.6. Given that three of the arrows in the above diagram are equivalences, we conclude that the fourth one, $Q_n(\Gamma T_{\mathbf{a}}T_{\mathbf{a}}^k C) \rightarrow Q_n(T_{\mathbf{a}}T_{\mathbf{a}}^k C)$, is also a weak equivalence, which proves that $I/I^n(C)$ and $Q_n(B_{\ast}^{\mathbf{a}}\Gamma C)$ have equivalent simplices. We conclude that the map $I/I^{n+1}(C) \rightarrow I/I^n(C)$ is a weakly equivalent to a fibration which, on the level of $(k+1)$ -simplices is given by projections

$$\mathbf{a}(\mathbf{1}) \wedge T_{\mathbf{a}}^k(C) \vee \cdots \vee \mathbf{a}(\mathbf{n}) \wedge_{\Sigma_n} (T_{\mathbf{a}}^k(C))^{\wedge n} \rightarrow \mathbf{a}(\mathbf{1}) \wedge T_{\mathbf{a}}^k(C) \vee \cdots \vee \mathbf{a}(\mathbf{n}-\mathbf{1}) \wedge_{\Sigma_{n-1}} (T_{\mathbf{a}}^k(C))^{\wedge n-1}.$$

Thus, for all $n \geq 1$, $I^n/I^{n+1}(C)$ is equivalent to a simplicial A -module with $\mathbf{a}(n) \wedge_{\Sigma_n} (T_{\mathbf{a}}^k(C))^{\wedge n}$ for $(k+1)$ -simplices. In particular, the simplices of I/I^2 are $\mathbf{a}(\mathbf{1}) \wedge T_{\mathbf{a}}^k(C)$. Furthermore, note that by Lemma 5.6, the orbits and homotopy orbits with respect to symmetric groups Σ_n of the objects involved (i.e. of operadic powers of $T_{\mathbf{a}}^k(C)$) are equivalent. Consequently, given that \mathbf{a} is primitively generated and hence the multiplication maps $\mathbf{a}(n) \wedge \mathbf{a}(\mathbf{1}) \wedge \cdots \wedge \mathbf{a}(\mathbf{1}) \rightarrow \mathbf{a}(n)$ are equivalences (see the comments following Definition 5.3), we get that $I^n/I^{n+1}(C) \simeq \mathbf{a}(n) \wedge_{h\Sigma_n} \bigwedge^n (I/I^2(C))$ as desired. □

We return to the question of ‘approximating’ towers. The following proposition makes our previously used language precise. Roughly speaking, it states that under favorable conditions, the tower of functors $\{I/I^n\}$ converges to the forgetful functor $U_{\mathbf{a}}$.

Proposition 5.7. *Let A be a connective commutative S -algebra, and \mathbf{a} a primitively generated operad in the category of A -modules. Then for every 0 -connected \mathbf{a} -algebra C , the natural map*

$$\phi : U_{\mathbf{a}}(\Gamma C) \simeq U_{\mathbf{a}}(\Gamma B_{\ast}^{\mathbf{a}}\Gamma C) \rightarrow \operatorname{holim}_n I/I^n(C)$$

is a weak equivalence.

Proof. We begin by observing that $(\Gamma B_{\ast}^{\mathbf{a}}\Gamma C)^{\wedge n}$ is at least $n-1$ -connected. Indeed, since A is connective, the Cellular Approximation Theorem [Chapter 3 of [7]] allows us to functorially replace $\Gamma B_{\ast}^{\mathbf{a}}\Gamma C$ by a weakly equivalent CW A -module M with no cells in dimensions below one. By properties listed in Section 9,

the derived smash powers of $\Gamma B_*^{\mathbf{a}}\Gamma C$ are defined on the point set level. Hence, we have that $(\Gamma B_*^{\mathbf{a}}\Gamma C)^{\wedge n}$ is equivalent to $M^{\wedge n}$. (Once again, this last statement is not immediate since $\Gamma B_*^{\mathbf{a}}\Gamma C$ is not necessarily cofibrant as an A -module.) Observe that $M^{\wedge n}$ (and consequently $(\Gamma B_*^{\mathbf{a}}\Gamma C)^{\wedge n}$) is $(n-1)$ -connected, since M has no cells in dimensions below 1, and thus $M^{\wedge n}$ has no cells in dimensions less than n .

Hence, we are allowed to conclude that the connectivity of the maps $U_{\mathbf{a}}(\Gamma B_*^{\mathbf{a}}\Gamma C) \rightarrow I/I^n(C)$ increases with n , producing a Mittag-Leffler system (see [4]), and thus implying the claim of the proposition. \square

As we will see in the next section, Proposition 5.5 is a critical ingredient for obtaining a splitting result for \mathbf{a} -algebras. Of course, in that proposition, the assumption that the operad \mathbf{a} be primitively generated is also used in an essential way. In fact, without that assumption, the tower I/I^n that we constructed to approximate the forgetful functor $U_{\mathbf{a}}$ from the category of \mathbf{a} -algebras to A -modules does not enjoy the properties required to develop our theory.

However, as we will show momentarily, the Goodwillie tower of the forgetful functor does. In fact, for a primitively generated operad \mathbf{a} , the tower $\{I/I^n\}$ is the Goodwillie tower of the forgetful functor. In other words, the condition on the operad simply allowed us to provide a nice description of the Goodwillie tower in terms of functors I/I^n .

The general version of our result still requires a condition on the operad, however it is very mild as nearly all naturally occurring operads satisfy it.

Proposition 5.8. *Let the operad \mathbf{a} be such that for all $n > 0$ the operadic multiplication maps*

$$\mathbf{a}(n) \wedge \mathbf{a}(1) \wedge \cdots \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(n)$$

are equivalences. Then for every \mathbf{a} -algebra C , the n 'th layer D_n of the Goodwillie tower of the forgetful functor $U_{\mathbf{a}}$ can be described as

$$D_n U_{\mathbf{a}}(C) \simeq \mathbf{a}(n) \wedge_{h\Sigma_n} (D_1 U_{\mathbf{a}}(C))^{\wedge n}.$$

Proof. We employ the same strategy as when proving Proposition 5.5. In other words, instead of the Goodwillie tower $P_n U_{\mathbf{a}}(\Gamma C)$ consider the equivalent tower $P_n U_{\mathbf{a}}(\Gamma B_*^{\mathbf{a}}\Gamma C)$. As before, we can strip the outer cofibrant replacement functor Γ . Indeed, let $\gamma : Y \rightarrow T_{\mathbf{a}}^k \Gamma C$ be a cell A -module approximation and consider the commutative square

$$\begin{array}{ccc} P_n U_{\mathbf{a}}(\Gamma T_{\mathbf{a}} Y) & \longrightarrow & P_n U_{\mathbf{a}}(\Gamma T_{\mathbf{a}} T_{\mathbf{a}}^k \Gamma C) \\ \downarrow & & \downarrow \\ P_n U_{\mathbf{a}}(T_{\mathbf{a}} Y) & \longrightarrow & P_n U_{\mathbf{a}}(T_{\mathbf{a}} T_{\mathbf{a}}^k \Gamma C). \end{array}$$

The left vertical and the top horizontal arrows are equivalences by the same argument as in Proposition 5.5. To see that the bottom horizontal map is an equivalence, note that since $T_{\mathbf{a}}$ is coproduct preserving, $P_n U_{\mathbf{a}}(T_{\mathbf{a}}Y) \simeq P_n(U_{\mathbf{a}}T_{\mathbf{a}})(Y)$. Consequently,

$$P_n U_{\mathbf{a}}(T_{\mathbf{a}}Y) = P_n[U_{\mathbf{a}}(\bigoplus_{i=1}^{\infty} \mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i})] \simeq P_n[U_{\mathbf{a}}(\bigoplus_{i=1}^{\infty} \mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i})] \simeq \bigoplus_{i=1}^{\infty} P_n(\mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}).$$

Here we have used the fact that Y is a cell A -module (and thus orbits are equivalent to homotopy orbits in the above setup), and that the n 'th Taylor polynomial P_n commutes with coproducts of functors. Now note that $\mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}$ is an n -homogeneous functor, i.e. $P_n(\mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}) \simeq *$ if $n < i$, and $P_n(\mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}) \simeq \mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}$ for $n \geq i$. Thus,

$$P_n U_{\mathbf{a}}(T_{\mathbf{a}}Y) \simeq \bigoplus_{i=1}^{\infty} P_n U_{\mathbf{a}}(\mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}) \simeq \bigoplus_{i=1}^n \mathbf{a}(i) \wedge_{h\Sigma_i} Y^{\wedge i}.$$

Similarly,

$$P_n U_{\mathbf{a}}(T_{\mathbf{a}}T_{\mathbf{a}}^k \Gamma C) \simeq \bigoplus_{i=1}^n \mathbf{a}(i) \wedge_{h\Sigma_i} (T_{\mathbf{a}}^k \Gamma C)^{\wedge i},$$

except here since $T_{\mathbf{a}}^k \Gamma C$ is not necessarily a cell A -module, we make use of Lemma 5.6 and properties listed in Section 9. Now Lemma 5.6 implies that the bottom horizontal arrow of the diagram is a weak equivalence. Hence, so is the right vertical map, thus proving that we may strip away the outer functor Γ . Consequently, it suffices to prove the proposition for free \mathbf{a} -algebras only. The computations for the free case though are identical to those performed at the end of the proof of Proposition 5.5. □

We conclude this section by pointing out that Proposition 5.7 has its obvious analogue in this generalized set up as well.

Proposition 5.9. *Let A be a connective commutative S -algebra, and \mathbf{a} an operad in the category of A -modules, satisfying the condition of Proposition 5.8. Then for every 0-connected \mathbf{a} -algebra C , the natural map*

$$\phi : U_{\mathbf{a}}(\Gamma C) \rightarrow \operatorname{holim}_n P_n U_{\mathbf{a}}(\Gamma C)$$

is a weak equivalence.

Proof. The proof of Proposition 5.7 works here verbatim if we replace $I/I^{n+1}(C)$ by $P_n U_{\mathbf{a}}(\Gamma C)$. □

6 Algebras over Operads and Splitting

6.1 On Splitting the Forgetful Functor

In this subsection we address the question of detecting free objects in the category of \mathbf{a} -algebras. More precisely, given an \mathbf{a} -algebra C , when is it equivalent to $T_{\mathbf{a}}X$ for some A -module X ? Here, as before, \mathbf{a} is an operad arising from simplicial sets.

Theorem 6.1. *Let the operad \mathbf{a} be primitively generated. As before, let $U_{\mathbf{a}} : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$ be the forgetful functor from the category of \mathbf{a} -algebras to the category of A -modules. Furthermore, suppose the \mathbf{a} -algebra C is such that the natural map $U_{\mathbf{a}}(C) \rightarrow I/I^2(C)$ has a section $\phi : I/I^2(C) \rightarrow U_{\mathbf{a}}(C)$ in the category of A -modules. Then we have a weak equivalence of \mathbf{a} -algebras*

$$\operatorname{holim}_n I/I^n(C) \simeq \prod \mathbf{a}(n) \wedge_{h\Sigma_n} [I/I^2(C)]^{\wedge n}.$$

Proof. Note that by applying the functor $T_{\mathbf{a}}$ to the section $\phi : I/I^2(C) \rightarrow U_{\mathbf{a}}(C)$ and composing the resulting \mathbf{a} -algebra map with the multiplication map μ with which the algebra C is equipped, we get a composite map α of \mathbf{a} -algebras

$$\alpha : T_{\mathbf{a}}[I/I^2(C)] \xrightarrow{T_{\mathbf{a}}\phi} T_{\mathbf{a}}[U_{\mathbf{a}}(C)] \xrightarrow{\mu} C.$$

Consider the map $I/I^n(\alpha)$:

$$I/I^n(T_{\mathbf{a}}[I/I^2(C)]) \longrightarrow I/I^n(C).$$

We claim that for all $n > 1$, this map is a weak equivalence. We induct on n .

For $n = 2$, the left-hand side is equivalent to $I/I^2(C)$, as clearly is the right-hand side. Moreover the map $I/I^2(\alpha)$ is an equivalence since by assumption the composite $I/I^2(C) \rightarrow U_{\mathbf{a}}(C) \rightarrow I/I^2(C)$ is an equivalence.

Next assume that the claim is true for $n = k$, i.e that $I/I^k(\alpha) : I/I^k(T_{\mathbf{a}}[I/I^2(C)]) \rightarrow I/I^k(C)$ is an equivalence, and consider the following diagram. Observe that the diagram is commutative since all the vertical maps are induced by α .

$$\begin{array}{ccccc} I^k/I^{k+1}(T_{\mathbf{a}}[I/I^2(C)]) & \longrightarrow & I/I^{k+1}(T_{\mathbf{a}}[I/I^2(C)]) & \longrightarrow & I/I^k(T_{\mathbf{a}}[I/I^2(C)]) & (13) \\ \downarrow & & \downarrow & & \downarrow & \\ I^k/I^{k+1}(C) & \longrightarrow & I/I^{k+1}(C) & \longrightarrow & I^k(C) & \end{array}$$

The right vertical map is a weak equivalence by inductive hypothesis. The source and the target of the left vertical map are both equivalent to $\mathbf{a}(k) \wedge_{h\Sigma_k} \bigwedge^k I/I^2(C)$ by Proposition 5.5. Moreover, the proof of

Proposition 5.5 implies that the diagram

$$\begin{array}{ccc} \mathbf{a}(k) \wedge_{h\Sigma_k} \bigwedge^k I/I^2(T_{\mathbf{a}}[I/I^2(C)]) & \xrightarrow{\simeq} & I^k/I^{k+1}(T_{\mathbf{a}}[I/I^2(C)]) \\ \downarrow & & \downarrow \\ \mathbf{a}(k) \wedge_{h\Sigma_k} \bigwedge^k I/I^2(C) & \xrightarrow{\simeq} & I/I^k(T_{\mathbf{a}}[I/I^2(C)]) \end{array}$$

commutes up to a weak equivalence. Here the left hand side map is induced by α . We already argued that I/I^2 applied to α is an equivalence, and hence the left vertical map is an equivalence. This implies that the right vertical map, or equivalently (given that it is the same map), the left vertical map of Diagram 13 is an equivalence. Hence the middle vertical map of Diagram 13 is also a weak equivalence, thus proving that for all $n > 1$,

$$I/I^n(\alpha) : I/I^n(T_{\mathbf{a}}[I/I^2(C)]) \longrightarrow I/I^n(C)$$

is a weak equivalence. Consequently,

$$\operatorname{holim}_n I/I^n(C) \simeq \operatorname{holim}_n I/I^n(T_{\mathbf{a}}[I/I^2(C)]) \simeq \prod_n \mathbf{a}(n) \wedge_{\Sigma_n} [I/I^2(C)]^{\wedge n}$$

□

Observe that whenever in addition to the hypotheses of this theorem, C and $T_{\mathbf{a}}C$ are complete. i.e. they are equivalent to the homotopy limit of their respective Goodwillie towers (for example if the \mathbf{a} -algebras C satisfies the connectivity assumptions of Proposition 5.7), this theorem implies that C is a free object in the category \mathbf{a} -algebras, thus answering the earlier posed question.

Remark 6.2. Naturally, Proposition 5.8 allows us to prove the analogue of this theorem for general operads. We do state the general version of the theorem for completeness of the picture and reference purposes, but do not present the proof here. Instead we comment that the proof of Theorem 6.1 works here nearly verbatim, once we replace $I/I^n(C)$ by $P_{n-1}U_{\mathbf{a}}(C)$, $I^n/I^{n+1}(C)$ by $D_nU_{\mathbf{a}}(C)$, and use Proposition 5.8 whenever Proposition 5.5 is used.

Theorem 6.3. *Let the operad \mathbf{a} be such that the operadic multiplications*

$$\mathbf{a}(n) \wedge \mathbf{a}(1) \wedge \cdots \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(n)$$

are equivalences. Furthermore, suppose the \mathbf{a} -algebra C is such that the natural derivative map $U_{\mathbf{a}}(C) \rightarrow D_1U_{\mathbf{a}}(C)$ has a section in the category of A -modules. Then we have a weak equivalence

$$\operatorname{holim}_n P_nU_{\mathbf{a}}(C) \simeq \prod_n \mathbf{a}(n) \wedge_{h\Sigma_n} [D_1U_{\mathbf{a}}(C)]^{\wedge n}.$$

Combining the results of this section with the main theorem of [2] yields an immediate corollary on the structure of $co-H$ -objects in the category of algebras over a fixed operad \mathbf{a} . We begin by recalling a definition from [2].

Definition 6.4. Let \mathcal{C} be a pointed model category, and let X be a cofibrant object in \mathcal{C} . We say that X is a $co-H$ -object of \mathcal{C} if there exists a comultiplication $\nabla : X \rightarrow X \vee X$ which is coassociative and counital up to homotopy,

The obvious relationship of $co-H$ -objects with the classical notion of quasi-Hopf algebras is what prompted the choice of terminology in the above definition.

Theorem 6.5 (Bauer-McCarthy, [2]). *Let F be a functor from \mathcal{C} to the category of spectra and let X be a $co-H$ -object of \mathcal{C} . Then rationally,*

$$\operatorname{holim}_n P_n F(X) \simeq \prod_{n \geq 0} D_n F(X).$$

In particular, applying this theorem to the forgetful functor $U_{\mathbf{a}} : \mathcal{C}_{\mathbf{a}} \rightarrow \mathcal{M}_A$ for a fixed operad \mathbf{a} as above, we conclude that if C is a $co-H$ -object of $\mathcal{C}_{\mathbf{a}}$, then the natural derivative map $U_{\mathbf{a}}(C) \rightarrow D_1 U_{\mathbf{a}}(C)$ has a section. Thus, the following corollary is immediate from Theorem 6.1 and Theorem 6.3.

Corollary 6.6. *Let the operad \mathbf{a} be such that the operadic multiplications*

$$\mathbf{a}(n) \wedge \mathbf{a}(1) \wedge \cdots \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(n)$$

are equivalences. Furthermore, suppose the \mathbf{a} -algebra C is a $co-H$ -object of $\mathcal{C}_{\mathbf{a}}$. Then rationally we have a weak equivalence

$$\operatorname{holim}_n P_n U_{\mathbf{a}}(C) \simeq \prod_n \mathbf{a}(n) \wedge_{h\Sigma_n} [D_1 U_{\mathbf{a}}(C)]^{\wedge n}.$$

If in addition, \mathbf{a} is primitively generated, then rationally

$$\operatorname{holim}_n P_n U_{\mathbf{a}}(C) \simeq \operatorname{holim}_n I/I^n(C) \simeq \prod_n \mathbf{a}(n) \wedge_{\Sigma_n} [I/I^2(C)]^{\wedge n}.$$

Furthermore, in [2] the authors extend the above Theorem 6.5 to the integral setting as follows.

Theorem 6.7. *Let F be a homotopy functor from \mathcal{C} to the category of spectra and let X be a cocommutative $co-H$ -object of \mathcal{C} . Then*

$$\operatorname{holim}_n P_n F(X) \simeq \prod_{n \geq 0} D_n F(X)$$

whenever the Tate cohomology

$$\operatorname{Tate}^n(F; X) \stackrel{\text{def}}{=} \operatorname{cofiber}[(D_1^{(n)} cr_n F(X))_{h\Sigma_n} \rightarrow (D_1^{(n)} cr_n F(X))^{h\Sigma_n}]$$

vanishes for all n .

Just as above, we combine this result with Theorem 6.1 and Theorem 6.3 to produce a corollary analogous to Corollary 6.6, which we state here for completeness and reference purposes.

Corollary 6.8. *Let the operad \mathbf{a} be such that the operadic multiplications*

$$\mathbf{a}(n) \wedge \mathbf{a}(1) \wedge \cdots \wedge \mathbf{a}(1) \rightarrow \mathbf{a}(n)$$

are equivalences. Furthermore, suppose the \mathbf{a} -algebra C is a cocommutative co- H -object of $\mathcal{C}_{\mathbf{a}}$. Then whenever the Tate cohomology $\mathrm{Tate}^n(U_{\mathbf{a}}; C)$ vanishes for all n , we have a weak equivalence

$$\mathrm{holim}_n P_n U_{\mathbf{a}}(C) \simeq \prod_n \mathbf{a}(n) \wedge_{h\Sigma_n} [D_1 U_{\mathbf{a}}(C)]^{\wedge n}.$$

If in addition, \mathbf{a} is primitively generated, then

$$\mathrm{holim}_n P_n U_{\mathbf{a}}(C) \simeq \mathrm{holim}_n I/I^n(C) \simeq \prod_n \mathbf{a}(n) \wedge_{\Sigma_n} [I/I^2(C)]^{\wedge n}.$$

6.2 Some Classical Theorems

As an example we consider the case when the operad \mathbf{a} is an E_{∞} operad, in other words the category of \mathbf{a} -algebras is the category of commutative A -algebras. In this case, whenever the commutative A -algebra C is such that the conditions of the theorem are satisfied we get the identity

$$\mathrm{holim}_n I/I^n(C) \simeq \mathbb{P}_A[I/I^2(C)],$$

where \mathbb{P} is the symmetric algebra cotriple, i.e. $\mathbb{P}(M) \cong \bigoplus M^{\wedge n}/\Sigma_n$, with the smash products taken over A . Moreover, if C is the Topological Hochschild homology of the commutative A -algebra A , i.e. by a theorem of McClure, Schwänzl and Vogt ([21]), $C \simeq A \otimes S^1$, then $I/I^2(C)$ is equivalent to the suspension of André-Quillen homology, in other words $I/I^2(C) \simeq \Sigma \mathrm{TAQ}(A)$ (see e.g. [23]). Hence, under the connectivity conditions of Proposition 5.7, we get

$$\mathrm{THH}(A) \simeq \mathbb{P}_A[\Sigma \mathrm{TAQ}(A)].$$

The next obvious question is ‘For which algebras A are the conditions of the theorem satisfied for $\mathrm{THH}(A)$?’ This has been discussed in detail in [20]. Here we present the answer and refer to [20] for proofs. We start by recalling a couple of definitions from there.

Definition 6.9. The map of commutative S -algebras $C \rightarrow D$ is étale if $\mathrm{TAQ}(D|C)$ is contractible.

We say that $\{C \rightarrow C_{\alpha}\}_{\alpha \in \mathcal{I}}$ is an étale covering of C if

1. each map $C \rightarrow C_{\alpha}$ is étale, and
2. for each pair of C -modules $M \rightarrow N$ such that $M \wedge C_{\alpha} \rightarrow N \wedge C_{\alpha}$ has a section for all α , the map $M \rightarrow N$ itself has a section $N \rightarrow M$ with $N \rightarrow M \rightarrow N$ equivalent to the identity on M .

Definition 6.10. The map of algebras $f : R \rightarrow C$ is smooth if there is an étale covering $\{C \rightarrow C_\alpha\}_{\alpha \in \mathcal{I}}$ of C such that for each α there is a factorization

$$R \longrightarrow \mathbb{P}_R X \xrightarrow{\phi} C_\alpha,$$

where X is a cell R -module and $\mathbb{P}_R X$ is the free commutative R -algebra generated by X , with ϕ étale.

The following theorem which answers our earlier posed question, has its analogue in discrete algebra, where it is commonly referred to as Hochschild-Kostant-Rosenberg (HKR) theorem. For complete details in discrete case we refer to [16].

Theorem 6.11. (HKR theorem, [20]) *Let $f : R \rightarrow A$ be smooth in the category of connective S -algebras. Then the natural (derivative) map $THH(A|R) \rightarrow \Sigma TAQ(A|R)$ has a section in the category of A -modules which induces an equivalence of A -algebras:*

$$\mathbb{P}_A \Sigma TAQ(A|R) \xrightarrow{\simeq} THH(A|R).$$

Next we illustrate how the theorem of Leray on structure of commutative quasi Hopf algebras is a special case of Theorem 6.1. First we recall some basic constructions from the theory of Hopf algebras. The extensively used paper [22] of J.Milnor and J.Moore is our main reference.

Let K be a commutative ring, and A an augmented K -algebra. We denote by $I(A)$ the augmentation ideal of A , i.e. $I(A) \stackrel{def}{=} \ker[A \rightarrow K]$.

Definition 6.12. If A is an augmented algebra over K , let $Q(A) = K \otimes_A I(A)$. The elements of the K -module $Q(A)$ are called the indecomposables elements of A . Further, note that there is a natural exact sequence

$$I(A) \otimes I(A) \rightarrow I(A) \rightarrow Q(A) \rightarrow 0.$$

The following is listed as Theorem 7.5 in [22].

Theorem 6.13. (Leray) *If A is a connected commutative quasi Hopf algebra over the field K of characteristic zero and $X = Q(A)$, then if $f : X \rightarrow I(A)$ is a morphism of graded vector spaces such that the composition $X \xrightarrow{f} I(A) \rightarrow X$ is the identity morphism of X , then there is an isomorphism of algebras $\mathbb{P}X \rightarrow A$ induced by f , where $\mathbb{P}X$ is the polynomial algebra on X .*

Connectedness of A is assumed to guarantee that the augmentation filtration on A is complete.

Observe that a connected commutative quasi Hopf algebra is in particular a commutative algebra. Thus, by Theorem 6.1 (or more precisely the special case of $\mathbf{a} = \mathbf{e}_\infty$), the algebra A is equivalent to the symmetric

algebra on $I/I^2(A)$ provided the derivative map $A \rightarrow I/I^2(A)$ has a section. However, the existence of such a section is precisely the hypothesis of the Leray Theorem once we recall that the module of indecomposables $Q(A)$ is $I/I^2(A)$. Thus the Leray theorem is simply a special case of our Theorem 6.1.

7 Operads Induced by Triples

Earlier, we recalled a construction that associated a triple to the given operad. In particular, it was easily seen that the algebras over the operad are same as the algebras over the triple.

In this section we introduce a construction that produces operads out of triples in the category of spectra.

It is worth noting that Andrew Mauer-Oats is currently working independently on a similar construction for functors of spaces, which presents challenges different (and most likely more complex) than the ones we encountered here while working with functors of spectra.

The notions of cross effects $cr_n F$ of a functor $F : \mathcal{B} \rightarrow \mathcal{A}$ from a basepointed category \mathcal{B} to an additive category \mathcal{A} , and of the derivative (or linearization) functor $D_1 F$ are the main ingredients of our constructions. See Section 2 for a brief discussion on both.

Recall that roughly speaking, $D_1 F$ is the linear approximation of F . It comes equipped with a natural (derivative) map $F \rightarrow D_1 F$ which is an equivalence if F is already of degree 1 (or linear). In particular, $D_1 D_1 F$ is equivalent to $D_1 F$.

The following Chain Rule lemma is a restatement in our set up of Lemma 5.7 of [14].

Lemma 7.1 (Chain Rule). *Let F and G be endofunctors of the category of S -modules \mathcal{M}_S . Then*

$$D_1(F \circ G) \simeq D_1 F \circ D_1 G.$$

We recall our notation convention. For a functor F of n variables, we denote by $D_1^i F(X_1, \dots, X_n)$ the derivative of F obtained by holding all but i 'th variable constant. Also, we denote by $D_1^{(n)} F$ the multilinearized functor $D_1^n \cdots D_1^2 D_1^1 F$.

Let $F : \mathcal{M}_S \rightarrow \mathcal{M}_S$ be a triple in the category of S -modules. In other words we have a multiplication map $\mu : F \circ F \Rightarrow F$ and a unit map $\eta : id_{\mathcal{M}_S} \Rightarrow F$, that satisfy the usual associativity and unit diagrams. Define

$$\mathbf{a}_F(n) = D_1^{(n)} cr_n F(S).$$

Our next objective is to demonstrate that the collection $\{\mathbf{a}_F(n)\}$ forms an operad. Naturally, we need to produce the operadic multiplication maps.

First note that since $id_{\mathcal{M}_S} \xrightarrow{\cong} D_1 id_{\mathcal{M}_S}$, applying the derivative functor D_1 to the natural transformation η produces the unit map $S \rightarrow D_1 cr_1 F(S)$. In what follows, for simplicity, we are going to assume that F is reduced, i.e $cr_1 F = F$.

Now fix an object $X \in \mathcal{M}_S$ and consider the triple multiplication map $F \circ F(\vee_n X) \rightarrow F(\vee_n X)$. We apply the functor D_1^1 (i.e. the derivative is taken with respect to the first summand, while the others are held constant), and use the Chain Rule Lemma 7.1 to get

$$D_1^1(F \circ F(\vee_n X)) \simeq D_1 F \circ D_1^1 F(\vee_n X) \rightarrow D_1^1 F(\vee_n X).$$

Iterate this construction by successively applying $D_1^2, D_1^3, \dots, D_1^n$, to get

$$D_1 F \circ D_1^{(n)} F(\vee_n X) \rightarrow D_1^{(n)} F(\vee_n X). \quad (14)$$

Recall that $D_1 F$ is a reduced linear functor. Hence, for each $Y \in \mathcal{M}_S$, $D_1 F(Y) = D_1 F(S \wedge Y) \cong D_1 F(S) \wedge Y$. Consequently, noting that $D_1^{(n)} F(\vee_n X) \simeq D_1^{(n)} cr_n F(X)$, we can rewrite the above Equation 14:

$$D_1 F(S) \wedge D_1^{(n)} cr_n F(X) \rightarrow D_1^{(n)} cr_n F(X). \quad (15)$$

Thus taking $X = S$ in this equation, we get the multiplication maps $\mathbf{a}_F(1) \wedge \mathbf{a}_F(n) \rightarrow \mathbf{a}_F(n)$.

Before producing the general multiplication map, it is beneficial (for better clarity) to understand the maps $\mathbf{a}_F(2) \wedge \mathbf{a}_F(n) \wedge \mathbf{a}_F(m) \rightarrow \mathbf{a}_F(n+m)$ first. To that end, consider the composite map

$$F \circ [F(X_1 \vee \dots \vee X_n) \vee F(X_{n+1} \vee \dots \vee X_{n+m})] \rightarrow F \circ F(X_1 \vee \dots \vee X_{n+m}) \xrightarrow{\mu} F(X_1 \vee \dots \vee X_{n+m}), \quad (16)$$

where the first map is induced by the obvious inclusions $F(X_1 \vee \dots \vee X_n) \rightarrow F(X_1 \vee \dots \vee X_{n+m})$ and $F(X_{n+1} \vee \dots \vee X_{n+m}) \rightarrow F(X_1 \vee \dots \vee X_{n+m})$. Applying $D_1^1, D_1^2, \dots, D_1^n$ successively, using the chain rule (Lemma 7.1) again, and setting $X_1 = \dots = X_{n+m} = X$, we get

$$D_1^1 F[D_1^{(n)} F(\vee_n X) \vee F(\vee_m X)] \rightarrow D_1^{(n)} F(\vee_{n+m} X), \quad (17)$$

where on the right the derivatives are taken with respect to the first n copies of X . Here we are treating the leftmost F of Equation 16 as a functor of two variable in the obvious way, consequently, the superscript 1 in $D_1^1 F$ on the left of the Equation 17 is a reflection of the fact that all n derivatives are taken over the first component $F(X_1 \vee \dots \vee X_n)$. Further applying $D_1^{n+1}, \dots, D_1^{n+m}$, using the fact that $D_1^{(l)} F(X_1 \vee \dots \vee X_l) \cong D_1^{(l)} cr_l F(X_1, \dots, X_l)$ for all $l > 1$, and continuing to set $X_1 = \dots = X_{n+m} = X$, we get a map

$$D_1^{(2)} cr_2 F[D_1^{(n)} cr_n F(X), D_1^{(m)} cr_m F(X)] \rightarrow D_1^{(n+m)} cr_{n+m} F(X). \quad (18)$$

Here we employed the fact that the last m derivatives are taken over the second component $F(X_{n+1} \vee \dots \vee X_{n+m})$ of the left hand side of Equation 16.

Recall that $D_1^{(2)}cr_2F$ is linear in each of its variables. Hence, for S -modules M and N , we have an identity

$$D_1^{(2)}cr_2F(M, N) \cong D_1^{(2)}cr_2F(S, S) \wedge M \wedge N = D_1^{(2)}cr_2F(S) \wedge M \wedge N.$$

Thus, Equation 18 can be rewritten as

$$D_1^{(2)}cr_2F(S) \wedge D_1^{(n)}cr_nF(X) \wedge D_1^{(m)}cr_mF(X) \rightarrow D_1^{(n+m)}cr_{n+m}F(X). \quad (19)$$

Specializing to $X = S$, we get the desired multiplication map $\mathbf{a}_F(2) \wedge \mathbf{a}_F(n) \wedge \mathbf{a}_F(m) \rightarrow \mathbf{a}_F(n+m)$.

The general case of the map $\mathbf{a}_F(k) \wedge \mathbf{a}_F(j_1) \wedge \cdots \wedge \mathbf{a}_F(j_k) \rightarrow \mathbf{a}_F(j_1 + \cdots + j_k)$ is analogous in spirit. Similar to the previous case, we employ the map

$$\begin{aligned} F \circ [F(X_1 \vee \cdots \vee X_{j_1}) \vee \cdots \vee F(X_{j_1+\cdots+j_{k-1}+1} \vee \cdots \vee X_{j_1+\cdots+j_k})] &\rightarrow F \circ F(X_1 \vee \cdots \vee X_{j_1+\cdots+j_k}) \\ &\rightarrow F(X_1 \vee \cdots \vee X_{j_1+\cdots+j_k}) \end{aligned}$$

induced by evident inclusions $X_{j_1+\cdots+j_{l-1}+1} \vee \cdots \vee X_{j_1+\cdots+j_l} \rightarrow X_1 \vee \cdots \vee X_{j_1+\cdots+j_k}$ (with $l \leq k$). Applying the sequence of D_1^i 's and setting $X_j = X$ for all j , we get the analogue of Equation 18:

$$D_1^{(k)}cr_kF[D_1^{(j_1)}cr_{j_1}F(X), \cdots, D_1^{(j_k)}cr_{j_k}F(X)] \rightarrow D_1^{(j_1+\cdots+j_k)}cr_{j_1+\cdots+j_k}F(X). \quad (20)$$

Finally, noting that $D_1^{(k)}cr_kF$ is linear in each of its k variables, we rewrite Equation 20 as

$$D_1^{(k)}cr_kF(S) \wedge D_1^{(j_1)}cr_{j_1}F(X) \wedge \cdots \wedge D_1^{(j_k)}cr_{j_k}F(X) \rightarrow D_1^{(j_1+\cdots+j_k)}cr_{j_1+\cdots+j_k}F(X), \quad (21)$$

to produce the general case of the multiplication map by setting $X = S$.

Theorem 7.2. *With the above defined multiplication, the collection $\{\mathbf{a}_F(n) = D_1^{(n)}cr_nF(S)\}_{n=0}^\infty$ forms an operad.*

Proof. We need to show that the associativity (Definition 4.1, part (a)), unital (part (b)) and equivariance (part (c)) diagrams commute for our operadic multiplication. We will use the same indexing as in Definition 4.1.

To prove the associativity consider the following object:

$$F \circ [F \circ (F(\vee_{i_1} X) \vee \cdots \vee F(\vee_{i_{j_1}} X)) \vee \cdots \vee F \circ (F(\vee_{i_{g_{k-1}+1}} X) \vee \cdots \vee F(\vee_{i_{g_k}} X))]. \quad (22)$$

There are two maps from this object to $F(\vee_{i_1+\cdots+i_{j_1}+\cdots+i_{g_{k-1}+1}+\cdots+i_{g_k}} X) = F(\vee_{i_1+\cdots+i_{g_k}} X)$.

Map 1. There is an evident map from Object 22 to

$$F \circ [F \circ F(\vee_{i_1+\cdots+i_{j_1}} X) \vee \cdots \vee F \circ F(\vee_{i_{g_{k-1}+1}+\cdots+i_{g_k}} X)], \quad (23)$$

given by inclusions of $F(\vee_{i_{g_{l-1}+1}} X) \vee \cdots \vee F(\vee_{i_{g_l}} X)$ into $F(\vee_{i_{g_{l-1}+1} + \cdots + i_{g_l}} X)$ for $l = 1, \dots, k$. Furthermore, noting that there is an evident inclusion map from $F \circ F(\vee_{i_1 + \cdots + i_{j_1}} X) \vee \cdots \vee F \circ F(\vee_{i_{g_{k-1}+1} + \cdots + i_{g_k}} X)$ to $F \circ F(\vee_{i_1 + \cdots + i_{j_1} + \cdots + i_{g_{k-1}+1} + \cdots + i_{g_k}} X)$, we get a map from Object 23 to

$$F \circ [F \circ F(\vee_{i_1 + \cdots + i_{j_1} + \cdots + i_{g_{k-1}+1} + \cdots + i_{g_k}} X)] = F \circ [F \circ F(\vee_{i_1 + \cdots + i_{g_k}} X)] \quad (24)$$

Finally, using the natural multiplication $F \circ (F \circ F) \Rightarrow F \circ F \Rightarrow F$, we get the desired map from Object 22 to $F(\vee_{i_1 + \cdots + i_{g_k}} X)$.

Map2. Now we begin by considering the map from Object 22 to

$$F \circ F \circ [F(\vee_{i_1} X) \vee \cdots \vee F(\vee_{i_{j_1}} X) \vee \cdots \vee F(\vee_{i_{g_{k-1}+1}} X) \vee \cdots \vee F(\vee_{i_{g_k}} X)], \quad (25)$$

given by the inclusion of $F \circ (F(\vee_{i_1} X) \vee \cdots \vee F(\vee_{i_{j_1}} X)) \vee \cdots \vee F \circ (F(\vee_{i_{g_{k-1}+1}} X) \vee \cdots \vee F(\vee_{i_{g_k}} X))$ into $F \circ [F(\vee_{i_1} X) \vee \cdots \vee F(\vee_{i_{j_1}} X) \vee \cdots \vee F(\vee_{i_{g_{k-1}+1}} X) \vee \cdots \vee F(\vee_{i_{g_k}} X)]$. Moreover, using an analogous inclusion we get a map from Object 25 to

$$(F \circ F) \circ [F(\vee_{i_1 + \cdots + i_{j_1} + \cdots + i_{g_{k-1}+1} + \cdots + i_{g_k}} X)] = (F \circ F) \circ [F(\vee_{i_1 + \cdots + i_{g_k}} X)]. \quad (26)$$

Similar to the previous case, the natural multiplication $(F \circ F) \circ F \Rightarrow F \circ F \Rightarrow F$, produces the second map from Object 22 to $F(\vee_{i_1 + \cdots + i_{g_k}} X)$.

Of course, our interest in these maps is explained by the fact that applying the multilinearization functor $D_1^{(i)}$ to the target and source and setting $X = S$, we get multiplication maps

$$\mathbf{a}_F(k) \wedge \left(\bigwedge_{s=1}^k \mathbf{a}_F(j_s) \right) \wedge \left(\bigwedge_{r=1}^j \mathbf{a}_F(i_r) \right) \rightarrow \mathbf{a}_F(i),$$

where the indexing is still the same as in Definition 4.1. Moreover, note that $D_1^{(i)}$ applied to Map 1 is the composition of the smash product of multiplications

$$\mathbf{a}_F(j_l) \wedge \mathbf{a}_F(i_{g_{l-1}+1}) \wedge \cdots \wedge \mathbf{a}_F(i_{g_l}) \rightarrow \mathbf{a}_F(i_{g_{l-1}+1} + \cdots + i_{g_l})$$

for $l = 1, \dots, k$, with the map

$$\mathbf{a}_F(k) \wedge \mathbf{a}_F(i_1 + \cdots + i_{g_1}) \wedge \cdots \wedge \mathbf{a}_F(i_{g_{k-1}+1} + \cdots + i_{g_k}) \rightarrow \mathbf{a}_F(i_1 + \cdots + i_{g_1} + \cdots + i_{g_{k-1}+1} + \cdots + i_{g_k}) = \mathbf{a}_F(i).$$

Observe that this composite corresponds to the left vertical map followed by the bottom horizontal map in the diagram of part (a) of Definition 4.1.

Similarly, $D_1^{(i)}$ of Map 2 is the composition of the maps

$$\mathbf{a}_F(k) \wedge \mathbf{a}_F(j_1) \wedge \cdots \wedge \mathbf{a}_F(j_k) \rightarrow \mathbf{a}_F(j_1 + \cdots + j_k) = \mathbf{a}_F(j)$$

and

$$\mathbf{a}_F(j) \wedge \mathbf{a}_F(i_1) \wedge \cdots \wedge \mathbf{a}_F(i_{g_1}) \wedge \cdots \wedge \mathbf{a}_F(i_{g_{k-1}+1}) \wedge \cdots \wedge \mathbf{a}_F(i_{g_k}) \rightarrow \mathbf{a}_F(i_1 + \cdots + i_{g_k}) = \mathbf{a}_F(i).$$

Consequently, the multilinearization of $D_1^{(i)}$ of Map 2 corresponds to the top horizontal arrow followed by the right vertical map in the diagram of part (a) of Definition 4.1.

To conclude the proof of the associativity, we note that the only difference between Map 1 and Map 2 (i.e. before we apply the multilinearization functor) is the order in which the three copies of the triple F are composed, and hence the associativity of the composition of the triple F implies that associativity of the operad \mathbf{a}_F .

To see that the unit diagrams for the operadic multiplication (part (b) of Definition 4.1) commute, consider the commutative diagram

$$\begin{array}{ccc} F(\vee_k S) & \xrightarrow{\cong} & F(\vee_k S) \\ \downarrow & \nearrow \mu & \\ F \circ F(\vee_k S) & & \end{array}$$

where the left vertical map is given by $\eta \circ id$ or $id \circ \eta$. In other words we have two diagrams here, and both of them commute since F is a unital triple. Observe that the commutative diagrams we get by applying $D_1^{(k)}$ to these diagrams, are precisely the unital diagrams of Definition 4.1, thus proving that our operadic multiplication is unital.

To prove that the operadic multiplication is equivariant we need to show that the two diagrams of part (c) of Definition 4.1 commute. Again we use the indexing and notation introduced there. Note that the left diagram is simply the multilinearization of the diagram

$$\begin{array}{ccc} F \circ (F(\vee_{j_1} S) \vee \cdots \vee F(\vee_{j_k} S)) & \longrightarrow & F \circ (F(\vee_{\sigma(j_1)} S) \vee \cdots \vee F(\vee_{\sigma(j_k)} S)) \\ \downarrow & & \downarrow \\ F \circ F(\vee_{j_1} S \vee \cdots \vee \vee_{j_k} S) & & F \circ F(\vee_{\sigma(j_1)} S \vee \cdots \vee \vee_{\sigma(j_k)} S) \\ \downarrow & & \downarrow \\ F(\vee_{j_1} S \vee \cdots \vee \vee_{j_k} S) & \longrightarrow & F(\vee_{\sigma(j_1)} S \vee \cdots \vee \vee_{\sigma(j_k)} S), \end{array}$$

hence, it is enough to show that this diagram commutes. However, the commutativity of this diagram is immediate by functoriality of F and the fact that $F \circ F \Rightarrow F$ is a natural transformation.

It remains to prove that the right diagram of part (c) of Definition 4.1 commutes. To ease the notation we will consider the special case $k = 1$. The general case is identical to the case $k = 1$. To keep track of the group action, we denote the sphere S by S_1, S_2, \dots, S_j , in other words, all S_i 's are equal to S and the

subscripting is to make the action of $\tau \in \Sigma_j$ transparent. Consider the diagram

$$\begin{array}{ccc} F \circ F(S_1 \vee \cdots \vee S_j) & \longrightarrow & F \circ F(S_{\tau(1)} \vee \cdots \vee S_{\tau(j)}) \\ \downarrow & & \downarrow \\ F(S_1 \vee \cdots \vee S_j) & \longrightarrow & F(S_{\tau(1)} \vee \cdots \vee S_{\tau(j)}). \end{array}$$

This diagram is commutative since $F \circ F \Rightarrow F$ is a natural transformation. The right equivariance diagram of part (c) of Definition 4.1 follows once we apply the multilinearization functor $D_1^{(j)}$ to this diagram. \square

Remark 7.3. In the above construction, as well as in Theorem 7.2, we considered triples in the category of S -modules \mathcal{M}_S . This was done simply as a matter of convenience, as everything in this section can be repeated verbatim for triples in the category \mathcal{M}_A of A -modules, where A is a cofibrant commutative S -algebra. In fact, the construction can be performed in any symmetric monoidal category that falls within the framework of [14], in other words, all symmetric monoidal categories where the identity functor is linear and for which the linearization functor D_1 satisfies the properties employed here, i.e. the chain rule property, the idempotency, and the correspondence of compositions to smash products (or whatever the symmetric monoidal operation happens to be).

The chain complexes $Ch(K)$ over a commutative ring K is an example of such a category.

Example 7.4. Here we present a few important examples in the category $Ch(K)$ of chain complexes over a field K of characteristic 0.

Commutative algebras. The symmetric algebra triple $S(X) = \bigoplus_n X^{\otimes n} / \Sigma_n$ produces the commutative algebras in the category $Ch(K)$. By [14] or [24], $D_1^{(n)} cr_n S(X) = X^{\otimes n}$. Hence, the operad \mathbf{a}_S is such that $\mathbf{a}_S(n) = K$. In other words, the operad induced by $S(X)$ is the e_∞ operad. In this and following examples we use the notation of [8] when referring to specific operads. See [8] for details on these operads.

Associative algebras. The associative algebras are algebras over the tensor algebra triple $T(X) = \bigoplus_n X^{\otimes n}$. By computations performed in Section 6 of [24], $D_1^{(n)} cr_n T(K) = K[\Sigma_n]$, implying that the associated operad \mathbf{a}_T is the operad e_1 .

Lie algebras. For the free Lie algebra triple L , by Section 7 of [24], the multilinearization of the cross effects $D_1^{(n)} cr_n L(K)$ is equivalent to the $n - 1$ -dimensional *Lie* representation of the symmetric group Σ_n . Consequently, the induced operad \mathbf{a}_L is the *Lie* operad.

n-Poisson algebras, $n \geq 2$. The free n -Poisson algebras are produced by the triple

$$P_n(X) = S(\Sigma^{1-n} L(\Sigma^{n-1} X)),$$

where S and L are the symmetric and Lie algebra triples respectively, and Σ is the suspension (shift) functor in the category of chain complexes. By Lemma 10.2 of [24],

$$D_1^{(k)} cr_k P_n(K) \cong \bigotimes_{j=1}^{k-1} H^* \left(\bigvee_j S^{n-1} \right),$$

where on the right hand side we have the tensor product of cohomologies (with coefficients in K) of the wedge of j copies of the $n - 1$ sphere S^{n-1} . Hence by a result of F.Cohen (Lemma 6.2 of [5]), the operad \mathbf{a}_{P_n} for $n \geq 2$, is equivalent to the homology of little n -cubes operad e_n , which in turn is isomorphic to the n -Poisson operad p_n .

Remark 7.5. Another simple observation is that this mechanism of producing operads out of triples is in fact a functor from the category of triples \mathcal{T} to the category of operads \mathcal{O} . Indeed, if $\tau : F \rightarrow G$ is a natural transformation which is a morphism of triples, then it induces the necessary maps $D_1^{(n)} cr_n F(S) \rightarrow D_1^{(n)} cr_n G(S)$, which are compatible with operadic multiplications, since τ respects the triple multiplications of F and G .

Moreover, note that if \mathbf{a} is an operad and $T_{\mathbf{a}}$ is the triple associated with it, then

$$D_1^{(n)} cr_n T_{\mathbf{a}}(S) = D_1^{(n)} [cr_n \bigoplus_{k=0}^{\infty} \mathbf{a}(k) \wedge_{h\Sigma_k} S^{\wedge k}] \cong \mathbf{a}(n). \quad (27)$$

To see this, recall that $D_1^{(n)} cr_n T_{\mathbf{a}}(X) \cong D_1^{(n)} T_{\mathbf{a}}(X \vee \cdots \vee X)$, where there are n copies of X on the right hand side. To keep better track, we will denote these n copies X_1, \dots, X_n . Consider the summand $\mathbf{a}(k) \wedge_{h\Sigma_k} (X_1 \vee \cdots \vee X_n)^{\wedge k}$ for a fixed k . Expanding this term, we get a new sequence of summands each of which has precisely k factors. Hence, if $k < n$ then from each summand at least one X_i is missing, i.e. that summand is constant with respect to X_i . Thus D_i applied to that summand is contractible. Consequently, $D_1^{(n)} \mathbf{a}(k) \wedge_{h\Sigma_k} (X_1 \vee \cdots \vee X_n)^{\wedge k}$ vanishes for all $k < n$. Now if $k > n$, then each of the summands has at least one X_j in degree 2 or higher. In other words, that summand is at least 2-multireduced (see Definition 3.1 of [14]) in that variable. Hence, by Proposition 3.2 of [14], $D_1^{(n)} \mathbf{a}(k) \wedge_{h\Sigma_k} (X_1 \vee \cdots \vee X_n)^{\wedge k}$ vanishes for all $k > n$ as well. Finally, consider the case $k = n$. By the same reasoning, all the summands with at least one X_i missing, vanish. Hence, there are precisely $n!$ surviving terms - one for each way to form a string of length n out of X_1, \dots, X_n without repeating any of X_i 's. Equation 27 follows.

In other words, if (by abuse of notation) we denote by \mathcal{O} the category of operads on the full subcategory of cofibrant objects in \mathcal{M}_S (as opposed on all of \mathcal{M}_S), we are allowed to conclude that the composite of functors $\mathcal{O} \rightarrow \mathcal{T} \rightarrow \mathcal{O}$ is equivalent to the identity functor, as it is straightforward to check that it is the identity map on morphisms as well. The same, of course, is not the case with the composite $\alpha : \mathcal{T} \rightarrow \mathcal{O} \rightarrow \mathcal{T}$, as it is a well known fact that not every triple arises from an operad. In fact, the question of description of

triples which are fixed under α is a splitting question, thus explaining our interest in above constructions in the context of this work. Indeed, if $F \in \mathcal{T}$ is such that $\alpha(F) \cong F$, then F is the triple associated to some operad \mathbf{a} , and hence $F(X) \simeq \bigoplus_n \mathbf{a}(n) \wedge_{h\Sigma_n} X^{\wedge n}$.

We discuss this next.

8 Triples and Splitting

It is the intent of this section to develop splitting criteria for triples $F : \mathcal{M}_S \rightarrow \mathcal{M}_S$ in the category of S -modules. The work leading up to this point suggests two natural approaches. First, in Section 3 we discussed the splitting of Goodwillie towers of functors landing in \mathcal{M}_S . Triples in \mathcal{M}_S , of course, are examples of such functors. Consequently, one could attempt to specialize the results of Section 3.

The second approach is suggested by splitting conditions for algebras over an operad (Section 6), since F produces an operad \mathbf{a}_F , and thus one can hope that the results of Section 6 can be applied whenever $F(X)$ is an \mathbf{a}_F -algebra for some X in \mathcal{M}_S .

Here we explore questions arising as a result of pursuing these two directions. In this section we assume that F is complete (unless explicitly specified otherwise), or that F is equivalent to the homotopy limit of its Goodwillie tower. To avoid double subscripting, we denote the triple associated with the operad \mathbf{a}_F by T_F instead of $T_{\mathbf{a}_F}$, which would be more consistent with the notation used throughout this work.

First, following on Definition V.2.1 of [7], we present a new piece of terminology.

Definition 8.1. For an operad \mathbf{a} (or a triple T) in the category \mathcal{M}_S of S -modules, we say that C is an \mathbf{a} -ring spectrum (or a T -ring spectrum) if it is equipped with multiplication maps $\mathbf{a}(n) \wedge_{\Sigma_n} C^{\wedge n} \rightarrow C$ (or $TC \rightarrow C$), such that the usual unital and associativity diagrams commute up to a weak equivalence.

Proposition 8.2. *Let $(F, \mu : F \circ F \Rightarrow F, \eta : Id \Rightarrow F)$ be a triple in the category of S -modules, such that for all X in \mathcal{M}_S , the Goodwillie tower of $F(X)$ splits. Then $F(X)$ is an \mathbf{a}_F -ring spectrum for all X .*

Proof. Since the tower of F splits for all X and F is complete, each map $p_n : F(X) \rightarrow P_n F(X)$ has a section up to a weak equivalence, and each P_n decomposes into the coproduct of layers, i.e. $P_n \simeq \vee_{i=1}^n D_i F(X)$. Consequently, for all n , we have a natural map $D_n F(X) \rightarrow F(X)$ which, up to a weak equivalence, is a section to the composite $F(X) \rightarrow P_n F(X) \rightarrow D_n F(X)$, where the second map is the evident projection. Moreover, by Proposition 2.5 $D_n F$ is naturally equivalent to $D_1^{(n)} cr_n F_{h\Sigma_n}$. Hence we have natural maps $D_1^{(n)} cr_n F(X) \rightarrow F(X)$.

Recall that $D_1^{(n)} cr_n F$ is a short hand notation for $D_1^{(n)} cr_n F(X, \dots, X)$, which is linear in each of its n variables. Thus,

$$D_1^{(n)} cr_n F(X) \simeq D_1^{(n)} cr_n F(S) \wedge X^{\wedge n}.$$

For more on this see Section 3 or [19]. Hence, we have natural maps $D_1^{(n)} cr_n F(S) \wedge_{\Sigma_n} X^{\wedge n} \rightarrow F(X)$ which by the universal property of coproducts induce a natural transformation (up to a weak equivalence)

$$\nu : T_F(X) = \bigoplus_{\bigoplus}^n D_1^{(n)} cr_n F(S) \wedge_{\Sigma_n} X^{\wedge n} \Rightarrow F(X). \quad (28)$$

Note that both the source and the target of ν are triples, and the map itself respects the triple multiplications. In other words the diagrams

$$\begin{array}{ccc} T_F \circ T_F(X) & \longrightarrow & T_F(X) \\ \downarrow & & \downarrow \\ F \circ F(X) & \longrightarrow & F(X) \end{array} \quad (29)$$

commute for all X (up to a weak equivalence). Indeed, recall that the triple multiplication of T_F is, in essence, given by multilinearization (with respect to all variables) of the map

$$\begin{aligned} F \circ [F(X_1 \vee \dots \vee X_{j_1}) \vee \dots \vee F(X_{j_1+\dots+j_{k-1}+1} \vee \dots \vee X_{j_1+\dots+j_k})] &\rightarrow F \circ F(X_1 \vee \dots \vee X_{j_1+\dots+j_k}) \\ &\rightarrow F(X_1 \vee \dots \vee X_{j_1+\dots+j_k}), \end{aligned}$$

which is the multiplication map of F . See Section 7 for details.

Thus, we have produced a natural (up to a weak equivalence) map of triples $T_F \Rightarrow F$ which, in turn, gives rise to a multiplication

$$T_F \circ F(X) \xrightarrow{\nu_{F(X)}} F \circ F(X) \xrightarrow{\mu} F(X). \quad (30)$$

We claim that this map makes $F(X)$ into an \mathbf{a}_F -ring spectrum. To see this, it remains to show that the associativity diagram

$$\begin{array}{ccccccc} TT(F(X)) & \longrightarrow & T(F(F(X))) & \longrightarrow & F(F(F(X))) & \xrightarrow{F(\mu)} & F(F(X)) \\ \downarrow = & & & & & & \downarrow \\ TT(F(X)) & \longrightarrow & T(F(X)) & \longrightarrow & F(F(X)) & \longrightarrow & F(X) \end{array} \quad (31)$$

commutes (up to a weak equivalence). Note that replacing X by $F(X)$ in the commutative Diagram 29, we have that the bottom horizontal arrow is equivalent to

$$TT(F(X)) \rightarrow FF(F(X)) \xrightarrow{\mu_{F(X)}} F(F(X)) \rightarrow F(X).$$

We rewrite Diagram 31 as

$$\begin{array}{ccccccc}
TT(F(X)) & \longrightarrow & T(F(F(X))) & \longrightarrow & F(F(F(X))) & \xrightarrow{F(\mu)} & F(F(X)) \\
\downarrow = & & & \nearrow \simeq & & & \downarrow \\
TT(F(X)) & \longrightarrow & FF(F(X)) & \xrightarrow{\mu_{F(X)}} & F(F(X)) & \longrightarrow & F(X),
\end{array}$$

where the commutativity of the left half is evident, and the right half commutes by associativity of the triple multiplication of F . □

Note that in the above proposition, $F(X)$ is equipped with two multiplications, namely $T_F \circ F(X) \rightarrow F(X)$ and $F \circ F(X) \rightarrow F(X)$, and they are compatible in the sense that the diagram

$$\begin{array}{ccc}
T_F \circ F(X) & \longrightarrow & F(X) \\
\downarrow & & \downarrow = \\
F \circ F(X) & \longrightarrow & F(X)
\end{array} \tag{32}$$

commutes (up to a weak equivalence). This is a consequence of the way the \mathbf{a}_F -ring spectrum multiplication of $F(X)$ was defined; see Equation 30.

Our immediate goal is to show that the converse of Proposition 8.2 also holds. First we make a few comments.

Suppose we have a triple F which happens to have a natural structure of an \mathbf{a}_F -ring spectrum via a multiplication map $m : T_F(F(X)) \rightarrow F(X)$, or equivalently, via a sequence of maps $D_1^{(n)} cr_n F(S) \wedge_{\Sigma_n} F(X)^{\wedge n} \rightarrow F(X)$. Recalling again that $D_1^{(n)} cr_n F$ is linear in each of its n variables, this maps produce a new set of morphisms

$$D_1^{(n)} cr_n F(F(X))_{\Sigma_n} \rightarrow F(X). \tag{33}$$

Furthermore, applying the multilinearized functor $D_1^{(n)} cr_n F_{\Sigma_n}$ to the unit $\eta : Id \Rightarrow F$ of the triple F , we get morphisms $D_1^{(n)} cr_n F(X)_{\Sigma_n} \rightarrow D_1^{(n)} cr_n F(F(X))_{\Sigma_n}$, which combine with Equation 33 to produce maps $D_1^{(n)} cr_n F(X)_{\Sigma_n} \rightarrow F(X)$. By universal property of coproducts, these induce a natural transformation $\nu : T_F(X) \rightarrow F(X)$.

This allows us to introduce a definition.

Definition 8.3. For any triple F , which is naturally an \mathbf{a}_F -ring spectrum, we say that the two algebra structures on $F(X)$ (given by $\mu : F \circ F(X) \rightarrow F(X)$ and $\nu : T_F \circ F(X) \rightarrow F(X)$) are compatible if Diagram 32 commutes.

where the left-most horizontal arrow is the inclusion of the degree 1 component $D_1F(F(X))$ of $T_F \circ F$ into $T_F \circ F$. Here one should be careful not to confuse $D_1F(F(X))$ with $D_1(F \circ F)(X)$, as the former is D_1F evaluated at $F(X)$, while the latter is the derivative of the functor $F \circ F$. Observe that the left square of Diagram 36 commutes since the right vertical arrow is the derivative of the left one. The right square commutes by compatibility of the two algebra structures on $F(X)$ - note that it is simply D_1 applied to the compatibility diagram. The map $D_1(T_F \circ F)(X) \rightarrow D_1(F \circ F)(X)$ is an equivalence since the derivatives D_1 of T_F and F are equivalent (recall the definition of T_F), and by compatibility of the algebra structures.

Consequently, the composite of all top horizontal arrows with the right-most vertical map is equivalent to the composition of the two right-most slanted maps $D_1F(F(X)) \rightarrow D_1F(D_1F(X)) \rightarrow D_1F(X)$ of Diagram 35.

Suppose for all $n \leq k$ we have that $P_nF(X)$ is equivalent to the coproduct of layers $D_1F(X) \vee \cdots \vee D_nF(X)$. Consider the diagram

$$\begin{array}{ccccc}
D_{k+1}F(X) \simeq D_1^{(k+1)}cr_{k+1}F(X)_{h\Sigma_{k+1}} & \xrightarrow{d_{k+1}} & P_{k+1}F(X) & \xrightarrow{q_{k+1}} & P_kF(X) \\
& \nearrow & \downarrow q_{k+1} & & \downarrow q_k \\
D_kF(X) \simeq D_1^{(k)}cr_kF(X)_{h\Sigma_k} & \xrightarrow{d_k} & P_kF(X) & \xrightarrow{q_k} & P_{k-1}F(X)
\end{array}$$

To see that $P_{k+1}F(X)$ splits we simply need to show that the composite

$$P_kF(X) \simeq P_{k-1}F(X) \vee D_1^{(k)}cr_kF(X)_{h\Sigma_k} \rightarrow F(X) \xrightarrow{p_{k+1}} P_{k+1}F(X) \xrightarrow{q_{k+1}} P_kF(X) \quad (37)$$

is equivalent to the identity map. Here the first map on the component $P_{k-1}F(X)$ exists because by inductive hypothesis, $P_{k-1}F(X)$ is equivalent to the coproduct of layers $D_1 \vee \cdots \vee D_{k-1}$, and the maps on layers are defined via Equation 34. As in Theorem 3.1, it is enough to prove that Map 37 is equivalent to the identity map on the component $D_1^{(k)}cr_kF(X)_{h\Sigma_k}$, because the identity on the component $P_{k-1}F(X)$ follows by inductive hypothesis. In other words, we need show that the composite

$$D_1^{(k)}cr_kF(X)_{h\Sigma_k} \rightarrow F(X) \xrightarrow{p_k} P_kF(X), \quad (38)$$

which is the restriction of Map 37 to $D_1^{(k)}cr_kF(X)_{h\Sigma_k}$, is equivalent to the map $D_1^{(k)}cr_kF(X)_{h\Sigma_k} \xrightarrow{d_k} P_kF(X)$ of the Goodwillie tower of $F(X)$. We form a diagram analogous to the one for the base case:

$$\begin{array}{ccccc}
D_1^{(k)}cr_kF(X)_{h\Sigma_k} & \longrightarrow & D_1^{(k)}cr_kF(F(X))_{h\Sigma_k} & \longrightarrow & F(X) \xrightarrow{p_k} P_kF(X) \\
\downarrow & & \downarrow & \nearrow P_k(\mu) & \\
P_kF(X) & \xrightarrow{P_kF(\eta)} & P_k(F \circ F)(X) & &
\end{array} \quad (39)$$

where the left map in the top horizontal row is induced by the unit $\eta : Id \Rightarrow F$, and the middle horizontal map is the \mathbf{a}_F -algebra map on $F(X)$. To describe the middle vertical map, observe that for all functors F and G , there is a natural transformation $P_k F(G(X)) \rightarrow P_k(F \circ G)(X)$ induced by natural maps $cr_t F(G(X_1, \dots, X_t)) \rightarrow cr_t(F \circ G)(X_1, \dots, X_t)$. These are of course consequences of the maps $G(X_1) \vee \dots \vee G(X_t) \rightarrow G(X_1 \vee \dots \vee X_t)$ resulting from the universal property of coproducts. Hence, we have a natural map $P_k F(F(X)) \rightarrow P_k(F \circ F)(X)$, and the middle vertical arrow is simply the composite $D_1^{(k)} cr_k F(F(X))_{h\Sigma_k} \rightarrow P_k F(F(X)) \rightarrow P_k(F \circ F)(X)$.

Observe that to complete the proof of the inductive step, it is enough to show that Diagram 39 commutes, since the composite of the left vertical arrow with the bottom horizontal and the slanted arrows is precisely the map $D_1^{(k)} cr_k F(X)_{h\Sigma_k} \xrightarrow{d_k} P_k F(X)$. The left square of Diagram 39 commutes by naturality of Goodwillie towers, and the commutativity of the right triangle is a consequence of the compatibility of the two algebra structures on $F(X)$. More precisely, it follows from the fact that, analogous to the case $n = 1$, the diagram

$$\begin{array}{ccccc} T_F(F(X)) & \xrightarrow{p_k} & P_k(T_F \circ F)(X) & \longrightarrow & P_k(F \circ F)(X) \\ \downarrow & & \downarrow & & \downarrow P_k(\mu) \\ F(X) & \xrightarrow{p_k} & P_k F(X) & \xrightarrow{=} & P_k F(X) \end{array}$$

is commutative due to compatibility of the algebra structures. \square

At the beginning of this section we required that the triple F be complete. Of course this was a rather critical assumption as it allowed us to construct maps into F whenever we had compatible morphisms into Taylor polynomials $P_n F$ of F . In Section 3 we observed that one way to remove this assumption is to replace F by its completion $P_\infty F$ in every statement. Naturally, we need to be more careful here as we have the additional structure of a triple on our functors, and in order to perform replacements we first need to verify that the completion $P_\infty F$ is also equipped with a triple multiplication.

9 Cell \mathbf{a} -algebras

In this section, we prove the technical results which were utilized in Sections 5 and 6. In essence, we lay the groundwork to overcome the problems created by the fact that $T_{\mathbf{a}} C$ is not a cofibrant A -module even when C is. First we state the main results again.

Theorem 9.1. *Let C be a cofibrant \mathbf{a} -algebra and $\gamma : Y \rightarrow T_{\mathbf{a}}^n C$ a cell A -module approximation.*

1. *Then the induced map*

$$\mathbf{a}(i) \wedge_{\Sigma_i} Y^{\wedge i} \rightarrow \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i}$$

is a weak equivalence for all $i > 0$.

2. The projection $\mathbf{a}(i) \wedge_{h\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i} \rightarrow \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}}^n C)^{\wedge i}$ is an equivalence for all n and i .

We follow the strategy employed by M.Basterra in [1] to treat the special case of the non-unital commutative algebra operad. She, in turn, relied heavily on [7]. We recall a pair of definitions from [1].

Definition 9.2. Let \mathcal{F}_A be the class of all A -modules of the form

$$A \wedge_S S \wedge_{\mathcal{I}} \mathcal{I}(i) \rtimes_G K,$$

where K any G spectrum indexed on the universe U^i that has the homotopy type of a G -CW spectrum for some subgroup G of the symmetric group Σ_i . $\mathcal{I}(i)$ is the linear isometries operad (see Section I of [7]).

What makes the class \mathcal{F}_A of particular interest is the following result, which is listed as Theorem 9.5 in [1]; it is a generalization of Theorem VII.6.7 of [7].

Theorem 9.3. For each M in \mathcal{F}_A let $\Gamma M \rightarrow M$ be a cell A -module approximation. Then for any finite collection $\{M_1, \dots, M_n\}$ of \mathcal{F}_A , the induced maps

$$\Gamma M_1 \wedge \dots \wedge \Gamma M_n \rightarrow M_1 \wedge \dots \wedge M_n$$

are weak equivalences of A -modules.

In other words, for objects in \mathcal{F}_A the derived smash product is defined on point set level.

Definition 9.4. (see Definition 9.6 of [1]) An extended cell is a pair of the form $(X \wedge B_+^n, X \wedge S_+^{n-1})$, where $n \geq 0$ and $A \wedge_S S \wedge_{\mathcal{I}} \mathcal{I}(i) \rtimes_G K$ for a G -spectrum K indexed on U^i and which has the homotopy type of a G -CW spectrum for some $G < \Sigma_i$.

An extended cell A -module is an A -module $M \operatorname{colim} M_i$, with $M_0 = *$ and M_n derived from M_{n-1} via a pushout of A -modules

$$\begin{array}{ccc} \bigvee_{\alpha} X_{\alpha} \wedge S_+^{n_{\alpha}-1} & \longrightarrow & M_{n-1} \\ \downarrow & & \downarrow \\ \bigvee_{\alpha} X_{\alpha} \wedge B_+^{n_{\alpha}} & \longrightarrow & M_n. \end{array}$$

We intend to show that cell \mathbf{a} -algebras are extended cell A -modules. For now, we list some of the key properties of extended cell modules which make our interest in them evident.

1. For an extended cell A -module M and a subgroup H of the symmetric group Σ_n , the operadic power $\mathbf{a}(n) \wedge_H M^{\wedge n}$ is in \mathcal{F}_A .

2. For an extended cell A -module M , a group $H < \Sigma_n$, and an H -simplicial set P , the projection

$$P \wedge_{hH} M^{\wedge n} \rightarrow P \wedge_H M^{\wedge n}$$

is a weak equivalence of spectra.

3. Let $Y \rightarrow M$ be a cell A -module approximation of the extended cell module M . Then for all $H < \Sigma_i$ and H -simplicial sets P ,

$$P \wedge_H Y^{\wedge i} \rightarrow P \wedge_H M^{\wedge i}$$

is a weak equivalence. Moreover, there is a cell A -module approximation $Z \rightarrow P \wedge_H M^{\wedge i}$ such that for all $G < \Sigma_j$ and G -simplicial sets Q ,

$$Q \wedge_G Z^{\wedge j} \rightarrow Q \wedge_G (P \wedge_H M^{\wedge i})^{\wedge j}$$

is a weak equivalence.

4. Let $M = P \wedge_H K^{\wedge k}$ and $N = Q \wedge_J L^{\wedge l}$ be A -modules for some extended cell A -modules K and L , groups $H < \Sigma_k$, $J < \Sigma_l$, and an H -simplicial set P and J -simplicial set Q . Then there is a cell A -module approximation $Z \rightarrow M \vee N$ such that for all Σ_i -simplicial sets T ,

$$T \wedge_{\Sigma_i} Z^{\wedge i} \rightarrow T \wedge_{\Sigma_i} (M \vee N)^{\wedge i}$$

is a weak equivalence.

Properties 1 and 2 are consequences of the proof of Theorem 9.8 of [1], Property 3 follows from the proof of Theorem 9.10 of [1], and Property 4 from Proposition 9.11 once we recall that the operad \mathbf{a} arises from an operad in simplicial sets.

The following (promised) result allows us to take advantage of nice homotopical properties of extended cell A -modules.

Lemma 9.5. *Let C be a cell \mathbf{a} -algebra. Then it is an extended cell A -module.*

Proof. Since C is a cell \mathbf{a} -algebra, it can be expressed as $C = \text{colim} M_i$, with $M_0 = *$ and M_i obtained from M_{i-1} as a pushout

$$\begin{array}{ccc} T_{\mathbf{a}}E & \longrightarrow & M_{i-1} \\ \downarrow & & \downarrow \\ T_{\mathbf{a}}CE & \longrightarrow & M_i, \end{array} \tag{40}$$

where E is a wedge sphere A -modules.

Consider the simplicial object $\beta_*(T_{\mathbf{a}}CE, T_{\mathbf{a}}E, M_{i-1})$ whose p -simplices are $T_{\mathbf{a}}CE \amalg (T_{\mathbf{a}}E)^{\amalg p} \amalg M_{i-1}$, where \amalg is the coproduct in the category of \mathbf{a} -algebras. The face and degeneracy operators are defined using the maps

$$\mu : T_{\mathbf{a}}CE \amalg T_{\mathbf{a}}E \rightarrow T_{\mathbf{a}} \quad \text{and} \quad \nu : T_{\mathbf{a}}E \amalg M_{i-1} \rightarrow M_{i-1},$$

which are induced by the maps in Diagram 40. See Definition VII.3.5 of [7] for details.

A simple comparison of face and degeneracy operators shows that we have an equivalence of simplicial objects $\beta_*(T_{\mathbf{a}}E, T_{\mathbf{a}}E, T_{\mathbf{a}}E)$ and $(T_{\mathbf{a}}E) \otimes I$, where I is the standard simplicial 1-simplex with $p+2$ p -simplices. Consequently,

$$\beta_*(T_{\mathbf{a}}CE, T_{\mathbf{a}}E, M_{i-1}) \simeq T_{\mathbf{a}}CE \amalg_{T_{\mathbf{a}}E} (T_{\mathbf{a}}E \otimes I) \amalg_{T_{\mathbf{a}}E} M_{i-1}.$$

Further, observe that the collapse map induced by $I \rightarrow \{pt\}$

$$T_{\mathbf{a}}CE \amalg_{T_{\mathbf{a}}E} (T_{\mathbf{a}}E \otimes I) \amalg_{T_{\mathbf{a}}E} M_{i-1} \rightarrow T_{\mathbf{a}}CE \amalg_{T_{\mathbf{a}}E} M_{i-1} = M_i$$

is an equivalence. This is an immediate consequence of the of a simple homeomorphism of based spaces:

$$CX \cup_X (X \wedge I_+) \rightarrow CX.$$

The detailed proof of this statement is identical to that of Proposition VII.3.8 of [7]. Thus,

$$M_i \simeq |\beta_*(T_{\mathbf{a}}CE, T_{\mathbf{a}}E, M_{i-1})|.$$

To complete the proof, we note that the argument used to prove Lemma VII.7.5 of [7] shows that the q 'th filtration of $\beta_*(T_{\mathbf{a}}CE, T_{\mathbf{a}}E, M_{i-1})$ (and by passage to colimits any cell \mathbf{a} -algebra) is an extended cell A -module. □

Our next objective is to show that $T_{\mathbf{a}}^k C$ is a sum of objects of the form $(P \wedge C^{\wedge n})_H$ where H is a subgroup of the symmetric group Σ_n and P is an H -simplicial set. Observe that if this is the case, then since C is an extended cell module by Lemma 9.5, Theorem 9.1 is immediate from Properties 1 - 4.

As it often happens, the case $k = 2$, i.e. $T_{\mathbf{a}}^2 C$, conveys the essence of the problem, so we describe it in detail.

$$\begin{aligned}
T_{\mathbf{a}}^2 C &= T_{\mathbf{a}}(T_{\mathbf{a}} C) = \bigoplus_{i=1}^{\infty} \mathbf{a}(i) \wedge_{\Sigma_i} (T_{\mathbf{a}} C)^{\wedge i} \\
&\cong \bigoplus_{n=1}^{\infty} \bigoplus_{l=n}^{\infty} \mathbf{a}(n) \wedge_{\Sigma_n} \left[\bigoplus_{i_1+\dots+i_n=l} \mathbf{a}(i_1) \wedge \dots \wedge \mathbf{a}(i_n) \wedge_{\Sigma_{i_1} \times \dots \times \Sigma_{i_n}} C^{\wedge l} \right] \\
&\cong \bigoplus_{n=1}^{\infty} \bigoplus_{l=n}^{\infty} \bigoplus_{i_1+\dots+i_n=l} [\mathbf{a}(n) \wedge_{\Sigma_n} \mathbf{a}(i_1) \wedge \dots \wedge \mathbf{a}(i_n) \wedge_{\Sigma_{i_1} \times \dots \times \Sigma_{i_n}} C^{\wedge l}] \\
&\cong \bigoplus_{n=1}^{\infty} \bigoplus_{l=n}^{\infty} \bigoplus_{i_1+\dots+i_n=l} [\mathbf{a}(n) \wedge \mathbf{a}(i_1) \wedge \dots \wedge \mathbf{a}(i_n) \wedge C^{\wedge l}]_{\Sigma_n \ltimes (\Sigma_{i_1} \times \dots \times \Sigma_{i_n})},
\end{aligned}$$

where $\Sigma_n \ltimes (\Sigma_{i_1} \times \dots \times \Sigma_{i_n})$ is the semi-direct product, and is consequently a subgroup of Σ_l . Thus, each summand is indeed of the form $(P \wedge C^{\wedge l})_H$ with $H < \Sigma_l$.

The general case follows by induction. Indeed, $T_{\mathbf{a}}^{k+1} C = T_{\mathbf{a}}^k(T_{\mathbf{a}} C)$. Assuming that $T_{\mathbf{a}}^k C$ is a sum of objects of form $(P \wedge C^{\wedge l})_H$, we need to understand $(P \wedge (T_{\mathbf{a}} C)^{\wedge l})_H$. Observe that

$$(T_{\mathbf{a}} C)^{\wedge l} = \bigoplus_{(j_1, \dots, j_l)} \mathbf{a}(j_1) \wedge \dots \wedge \mathbf{a}(j_l) \wedge_{\Sigma_{j_1} \times \dots \times \Sigma_{j_l}} C^{\wedge (j_1 + \dots + j_l)}$$

Consequently, each summand of $(P \wedge (T_{\mathbf{a}} C)^{\wedge l})_H$ is itself a sum of objects of the desired form.

References

- [1] M.Basterra *André-Quillen Cohomology of Commutative S-algebras*. J. Pure App. Algebra **144**, 111-143. 1999
- [2] K.Bauer and R.McCarthy *On Vanishing Tate Cohomology and decompositions in Goodwillie calculus* Preprint, 2003
- [3] C.Berger and I.Moerdijk *Axiomatic homotopy theory for operads* Preprint, 2002
<http://front.math.ucdavis.edu/math.AT/0206094>
- [4] A.K.Bousfield and D.M.Kan *Homotopy limits, completions and localizations*. Springer Lecture Notes in Mathematics. Vol.**304**. 1972
- [5] F.Cohen *The Homology of \mathcal{C}_{n+1} spaces, $n \geq 0$* in “The homology of iterated loop spaces”, Lecture Notes in Math, **533**, 207-351. 1976
- [6] S.Eilenberg and S.MacLane *On the Groups $H(\pi, n)$, II* Ann. of Math. **60**, 49-139. 1954

- [7] A.D.Elmendorf, I.Kriz, M.A.Mandell and J.P.May *Rings, Modules, and Algebras in Stable Homotopy Theory* Mathematical Surveys and Monographs. Vol.47. AMS.1996
- [8] E.Getzler and J.D.S.Jones *Operads, Homotopy Algebra, and Iterated Integrals for Double Loop Spaces* Preprint. 1993
- [9] V.A.Ginzburg and M.M.Kapranov *Koszul Duality for Operads* Duke Math.J. **76**, 203-272. 1994
- [10] P.G.Goerss and M.J.Hopkins *Moduli Spaces of Commutative Ring Spectra* Preprint. 2002
- [11] T.Goodwillie *Calculus I: The First Derivative of Pseudoisotopy Theory* K-theory **4**, 1-27. 1990
- [12] T. Goodwillie *Calculus II: Analytic Functors* K-theory **5** 295-332. 1992
- [13] T. Goodwillie *Calculus III: The Taylor Series of a Homotopy functor* Preprint
- [14] B.Johnson and R. McCarthy *Deriving Calculus with Cotriples*. To appear in Trans.Amer.Math.Soc., 2003
- [15] I. Kriz, and J.P.May *Operads, Algebras, Modules and Motives* Astérisque, **233**. 1995
- [16] J.-L.Loday. *Cyclic Homology* Grundlehren Math. Wiss., 301. Springer-Verlag, New York-Berlin, 1992.
- [17] A.Mauer-Oats *Algebraic Goodwillie calculus and a cotriple model for the remainder* Preprint, 2003
- [18] J.P.May *The Geometry of Iterated Loop Spaces*. Lecture Notes in Math., vol **271**, Springer Berlin, 1972
- [19] R.McCarthy *On n -excisive functors of module categories* Preprint
- [20] R.McCarthy and V.Minasian *HKR Theorem for Smooth S -algebras*. J.Pure App. Algebra, **185/1-3**, 239-258. 2003
- [21] J.McClure, R.Schwänzl and R.Vogt *$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra* J. Pure App. Algebra **121**, 137-159. 1997
- [22] J.W.Milnor and J.C.Moore *On the Structure of Hopf Algebras* Ann. of Math. (2) **81**, 211-264. 1965
- [23] V.Minasian *André-Quillen spectral sequence for THH* Topology Appl., vol **129**, 273-280. 2003
- [24] V.Minasian *Functional Equations and Their Related Operads* Preprint, 2003