On n-excisive functors of module categories

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Abstract: We give a new construction for the n-th Taylor polynomial, in the sense of Goodwillie calculus, for homotopy functors from spectra to spectra. We then use this model to classify n-excisive functors of module categories of functors with smash product (FSPs) by bi-modules of explicit FSPs.

Introduction: In [Cal3], T. Goodwillie constructs a Taylor tower for functors from spectra to spectra. The layers of this construction are called n-homogeneous functors and he classifies these by use of the n-th cross effects of a functor. In this way, the layers of the tower become reasonably computable by using an explicit algorithm applied to the n-th cross effect of a functor. Our first goal in this paper is to establish a generalization of this process to not only obtain the layers of the Tower, but the tower itself. It should be pointed out, however, that our construction only works for functors of spectra while Goodwillie's classification of the layers also works in more general settings like homotopy functors of spaces. The idea for this construction arose from a conversation with G. Arone when he explained to me the Taylor tower of $\Sigma^\infty\Omega^\infty$ (from spectra to spectra).

In order to state the first result, we need to make some definitions. Let $\mathcal{M}_n$ be the <u>dual</u> of the category of non-empty finite sets and surjective morphisms (one object $\{1,\ldots,n\}$ for each $n \in \mathbb{N}$). For a pointed space X we let $X^{\wedge *}$ be the functor from $\mathcal{M}_n$ to pointed spaces defined by $U \in \mathcal{M}_n \mapsto X^{\wedge U}$ and

$$f : V \to U \mapsto \; X^f : X^{\wedge U} \longrightarrow X^{\wedge V} \quad X^f(x_1 \wedge \ldots \wedge x_u) = (x_{f(1)} \wedge \ldots \wedge x_{f(v)})$$

(the map induced by diagonals). For F a functor from spectra to itself we define $\widehat{cr}^* F$—a functor from $\mathcal{M}_n$ to functors of spectra. For $U \in \mathcal{M}_n$, $\widehat{cr}^U F$ is equivalent to Goodwillie's $|U|$-cross effect but defined dually using products and homotopy cofibers (see section 1 for details). For F a homotopy functor of spectra which is stably n-excisive we prove:

Theorem (4.6):

$$P_n F(\ ) \simeq \operatorname{hocolim}_k \; Map_{\mathcal{M}_n}((S^k)^{\wedge *}, \widehat{cr}^* F(S^k \wedge\ )).$$

In the second part of the paper we use this result to give a classification of n-excisive functors of module categories. A functor with smash product (FSP) is a model for ring spectra which is very convenient for explicit constructions (for example, they are used to define the Hochschild homology of ring spectra in [Bö]). One can easily work with the category of modules of an FSP and they inherit many of the useful properties that chain

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complexes do over (ordinary, discrete) rings. In particular, the category of spectra we are using is equivalent to the category of modules over the universal FSP (whose spectrum is equivalent to $\Sigma^\infty S^0$). Let A and B be FSPs. For F a functor from A-modules to B-modules, we say that F satisfies the *limit axiom* if F commutes up to equivalence with directed homotopy limits.

Theorem (7.6): The homotopy category of n-excisive functors from A-modules to B-modules which satisfy the limit axiom is equivalent to the homotopy category of $B - L_nS[M_n(A)]$-bimodules. There is a pairing

$$(B-Mod-L_nS[M_n(A)]) \times (C-Mod-L_mS[M_m(B)]) \xrightarrow{\mu} (C-Mod-L_{mn}S[M_{mn}(A)])$$

which recovers the composition of n–excisive functors from A-modules to B-modules with m–excisive functors from B-modules to C-modules by this equivalence.

The FSP $L_nS[M_n(A)]$ is defined explicitly in section 7 but is essentially the FSP analogue of a subring of the monoid ring $\mathbf{Z}[M_n(R)]$ when R is a discrete ring and $M_n(R)$ is a monoid by multiplication.

The paper is organized as follows. After establishing some preliminaries we define $\widehat{cr}^*F$ in section 1. In section 2 we define L_nF for $n \geq 1$ using $\widehat{cr}^*$ and with these we define a tower associated to F. In section 3 we examine the tower $\{L_n(F)\}_{n\geq 0}$ in a special case. We use induction and the results of section 3 to show in section 4 that $F \to L_nF$ is universal with respect to n–excisive functors and hence deduce that L_nF is naturally equivalent to P_nF when F is stably n-excisive. In section 5 we recall some preliminaries about module categories over an FSP. In section 6 we prove:

Theorem (6.2): Let F and G be n-excisive functors of module categories which satisfy the limit axiom. If $\eta : F \to G$ is a natural transformation such that η_{A^n} is an equivalence then η_M is an equivalence for all A–modules M.

In section 7 we classify L_nF for F satisfying the limit axiom by using bi-modules of explicit FSPs. This is done, as it is in ordinary algebra, by establishing a natural transformation which is trivially an equivalence at A^n between two n-excisive functors which satisfy the limit axiom and then appealing to theorem (6.2).

We would like to thank Greg Arone for sharing his thoughts with us. His ideas were the key ingredients for getting this paper started. I would also like to thank Aarhus University for its hospitality–it was during a visit there that the ideas in this paper were clarified while having several useful conversations with Bjørn Dundas. Lastly, we want to thank Tom Goodwillie for teaching us how to work more effectively with cubical diagrams and higher order excisive functors.

0. Preliminaries and conventions

We will follow the basic conventions and terminology of [Cal1] and [Cal2]. We summarize what we use below and establish some notation.

A *space* is a compactly generated topological space and products and function spaces are formed in that category in the usual manner. Basepoints are always nondegenerate and an equivalence of spaces is a weak homotopy equivalence. Homotopy limits and homotopy colimits are defined as in [B-K]. We write the continuous category of based spaces as $\mathcal{S}p_*$.

For us, a *spectrum* is a sequence of based spaces $\{E_i | i \geq 0\}$ and based maps $E_i \to \Omega E_{i+1}$. A map of spectra is a collection of maps $E_i \to F_i$ which strictly preserves the structure maps. An Ω-*spectrum* is a spectrum such that the structure maps are all equivalences. We can associate to every spectrum an Ω-spectrum by setting $\Omega^\infty E_i = \text{hocolim}_j\ \Omega^j E_{i+j}$. An equivalence of Ω-spectra is a map which is an equivalence of spaces for each i and an equivalence of spectra is a map which induces an equivalence of the associated Ω-spectra.

Our convention is that every based space is (-1)–connected, path-connected spaces are 0–connected, etc. We say that a spectrum E is *k–connected* if for all i sufficiently large, E_i is $i+k$–connected. Thus, E k-connected implies that $\pi_n(E) = \text{colim}_j \pi_{n+j} E_j$ is zero for $n < k$ but not conversely. However, if $\pi_n(E) = 0$ for $n < k$ then E is equivalent to its k–connected cover and hence equivalent to a k–connected spectrum. A map of spectra is k–connected if its homotopy fiber is $(k-1)$–connected. We topologize the category of spectra as follows. Given two spectra E and F, we give $Hom(E, F)$ the (compactly generated) subspace topology of $\prod_{i=0}^n Hom(E_i, F_i)$. We let $\mathcal{S}pec$ be the continuous category of all spectra and $\mathcal{S}pec_b$ the continuous full subcategory of $\mathcal{S}pec$ determined by spectra equivalent to a k–connected spectra for some $k \in \mathbf{Z}$. For G a group, a *spectrum with G–action* is a spectrum E such that each E_i is a (pointed) G–space and the structure maps are G–equivariant (G–acts trivially on the loop component). These are not G–spectra (a more sophisticated notion) and our basics for handling spectra with G–action follow those of [MSRI] and are in appendix B.

For X a based space and E a spectrum, we let $X \wedge E$ be the new spectrum defined by $(X \wedge E)_i = X \wedge E_i$ with structure maps (we give their adjoints)

$$S^1 \wedge (X \wedge E_i) \cong X \wedge S^1 \wedge E_i \overset{1_X \wedge (adj)}{\longrightarrow} X \wedge E_{i+1}.$$

We let $Hom(X, E)$ be the new spectrum defined by $Hom(X, E)_i = Hom(X, E_i)$ with structure maps

$$Hom(X, E_i) \overset{Hom(X,\ )}{\longrightarrow} Hom(X, \Omega E_{i+1}) \cong \Omega Hom(X, E_{i+1}).$$

We observe that in general, $X \wedge$ is an endofunctor of $\mathcal{S}pec_b$ and $Hom(X,\ )$ is also if, for example, X is a finite CW-complex. Recall that for based spaces there is a natural homeomorphism $Hom(X \wedge Y, Z) \cong Hom(X, Hom(Y, Z))$ and similarly if E and F are spectra we have natural homeomorphisms

$$Hom(X \wedge Y, E) \cong Hom(X, Hom(Y, E))$$

$$Hom_{\mathcal{S}pec}(Z \wedge E, F) \cong Hom(Z, Hom_{\mathcal{S}pec}(E, F)).$$

Let $\mathcal{S}$ and $\mathcal{S}'$ each be one of $\mathcal{S}pec_b$ or $\mathcal{S}pec$. A *homotopy functor* F from $\mathcal{S}$ to $\mathcal{S}'$ is a continuous functor which preserves weak equivalences. We call F *reduced* if $F(*) = *$. The category

$$Func_0(\mathcal{S}, \mathcal{S}')$$

is the category of reduced homotopy functors with continuous natural transformations as morphisms.

For S a finite set, we let $\mathcal{P}(S)$ be the category of all subsets of S and maps inclusions. We let $\mathcal{P}_0(S)$ be the subcategory of $\mathcal{P}(S)$ consisting of non-empty subsets and $\mathcal{P}^1(S)$ the subcategory of proper subsets. An S-cube is a functor $\mathcal{X}$ from $\mathcal{P}(S)$ to spectra. For an S-cube $\mathcal{X}$, if the natural map

$$\mathcal{X}(\emptyset) \longrightarrow \mathrm{holim}_{\mathcal{P}_0(S)} \mathcal{X}|_{\mathcal{P}_0(S)}$$

is k-connected then we say $\mathcal{X}$ is k-Cartesian and Cartesian if this map is an equivalence. Dually, if the natural map

$$\mathrm{hocolim}_{\mathcal{P}^1(S)} \mathcal{X}|_{\mathcal{P}^1(S)} \longrightarrow X(S)$$

is k-connected then we say $\mathcal{X}$ is k-coCartesian and coCartesian if this map is an equivalence. We note that for spectra, a cube is Cartesian if and only if it is coCartesian. Similarly, an S cube of spectra is k-Cartesian if and only if it is a $(k+|S|-1)$-coCartesian cube. A cube is *strongly coCartesian* if every subcube is coCartesian and for spectra this is equivalent to being *strongly Cartesian* as well.

Definition (4.1-[Cal2]): A homotopy functor F is *stably n–excisive* if the following is true for some numbers c and κ:

$E_n(c, \kappa)$: If $\mathcal{X} : \mathcal{P}(S) \to \mathcal{S}pec$ is any strongly co-Cartesian $(n+1)$-cube such that for all $s \in S$ the map $\mathcal{X}(\emptyset) \to \mathcal{X}(s)$ is k_s–connected and $k_s \geq \kappa$, then the diagram $F(\mathcal{X})$ is $(-c + \Sigma k_s)$-Cartesian.

Definition: $Func_0^{st(n)}(\mathcal{S}, \mathcal{S}')$ is the subcategory of $Func_0(\mathcal{S}, \mathcal{S}')$ determined by functors which satisfy stable n-excision.

1. Properties of $\widehat{cr}^* F$

Throughout this section, let $\mathcal{S}$ and $\mathcal{S}'$ be any of the categories of pointed spaces ($\mathcal{S}p_*$) or bounded below spectra ($\mathcal{S}pec_b$) or spectra ($\mathcal{S}pec$). In order to define $L_n F$, we need to define several additional objects and establish several auxiliary lemmas. The actual definition is in (2.2).

Definition (1.1): Let U be a finite, non-empty set and $f : U \to \mathcal{S}$ a function. We define U^f to be the following U–cube of $\mathcal{S}$:

$$X \subseteq U \mapsto \prod_{x\in X} f(x) \qquad \left(\prod_{\emptyset} = *\right)$$

$$Y \subseteq X \mapsto \prod_{y\in Y} f(y) \stackrel{i}{\longrightarrow} \prod_{x\in X} f(x)$$

is given by inclusion using the base point.

Given a *surjective* map of sets $V \stackrel{\alpha}{\longrightarrow} U$, we let $U^f(\alpha)$ be the U-subcube of $V^{f\circ\alpha}$ defined by

$$U^f(\alpha)(X \subseteq U) = \prod_{v\in\alpha^{-1}(X)} f(\alpha(v)).$$

We let

$$\tilde{\alpha} : U^f \longrightarrow U^f(\alpha)$$

be the map of U–cubes defined by

$$\prod_{x\in X} f(x) \stackrel{\tilde{\alpha}(X)}{\longrightarrow} \prod_{v\in\alpha^{-1}(X)} f(\alpha(v))$$

$$(x_1, \ldots, x_t) \mapsto (x_{\alpha(1)}, \ldots, x_{\alpha(s)})$$

(the map induced by diagonals on spaces).

Definition (1.2): Let F be a functor from $\mathcal{S}$ to $\mathcal{S}'$ which is *reduced* ($F(*) = *$). Given a function $f : U \to \mathcal{S}$, we define

$$\widehat{cr}^f F = \text{hcofib } F(U^f).$$

Given a surjective map $\alpha : V \longrightarrow U$, we define

$$\widehat{\alpha} : \widehat{cr}^f F \longrightarrow \widehat{cr}^{f\circ\alpha} F$$

to be the natural composite

$$\text{hcofib } F(U^f) \stackrel{F(\tilde{\alpha})}{\longrightarrow} \text{hcofib } F(U^f(\alpha)) \stackrel{(inc)_*}{\longrightarrow} \text{hcofib } F(V^{f\circ\alpha})$$

where $(inc)_*$ is the natural transformation induced by the inclusion of the full subcategory generated by: $\{\alpha^{-1}(X) \in \mathcal{P}(V) | X \in \mathcal{P}(U)\}$.

Lemma (1.3): Given surjective set maps $W \stackrel{\beta}{\longrightarrow} V \stackrel{\alpha}{\longrightarrow} U$, then for any function f from U to $\mathcal{S}$: $\widehat{\alpha \circ \beta} = \widehat{\beta} \circ \widehat{\alpha}$.

Proof: Let $\tilde{\beta}'$ be the map of U–cubes from $U^f(\alpha)$ to $U^f(\alpha \circ \beta)$ defined by using diagonals. Using the naturallity of $(inc)_*$ we obtain the following commutative diagram which establishes the result.

$$
\begin{array}{ccccc}
\text{hcofib } F(U^f) & \xrightarrow{F(\tilde{\alpha})} & \text{hcofib } F(U^f(\alpha)) & \xrightarrow{(inc)_*} & \text{hcofib } F(V^{f\circ\alpha}) \\
& F(\widetilde{\alpha \circ \beta}) \searrow & \downarrow{\scriptstyle F(\tilde{\beta}')} & & \downarrow{\scriptstyle F(\tilde{\beta})} \\
& & \text{hcofib } F(U^f(\alpha \circ \beta)) & \xrightarrow{(inc)_*} & \text{hcofib } F(V^{f\circ\alpha}(\beta)) \\
& & & (inc)_* \searrow & \downarrow{\scriptstyle (inc)_*} \\
& & & & \text{hcofib } F(W^{f\circ\alpha\circ\beta})
\end{array}
$$

Definition (1.4): For $X \in \mathcal{S}$, we also write X for the function from $\{1\}$ to $\mathcal{S}$ defined by sending the unique element 1 to X. For U another non-empty finite set there is a unique surjective map α from U to $\{1\}$ and we define

$$\widehat{cr}^U F(X) = \widehat{cr}^{X\circ\alpha} F.$$

We define $\mathcal{M}$ to be the dual of the category of finite, non-empty sets and *surjective* morphisms of sets with one object $\{1, \dots, n\}$ for each $n \in \mathbb{N}$. We have defined a functor

$$Func_0(\mathcal{S}, \mathcal{S}') \xrightarrow{\widehat{cr}} Func(\mathcal{M}, Func_0(\mathcal{S}, \mathcal{S}'))$$

defined by sending a functor F to the functor which takes U to $\widehat{cr}^U F$.

Example: Let Id be the identity from pointed spaces to itself. Then

$$\widehat{cr}^U Id(X) = \bigwedge^U X.$$

For a surjective map $V \xrightarrow{\alpha} U$, $\widehat{\alpha}(x_1, \dots, x_n) = (x_{\alpha(1)}, \dots, x_{\alpha(t)})$ (the map induced from the diagonal embedding).

Given a functor F from $\mathcal{S}$ to $\mathcal{S}'$, we will write $tr(F)$ for the trivial constant functor from $\mathcal{M}$ to functors which takes every object to F and every morphism to the identity natural transformation. We now construct a natural transformation from tr to $\widehat{cr}$. For $U \in \mathcal{M}$ and $Z \in \mathcal{S}$, let $tr_Z(U)$ be the following U–cube:

$$X \subseteq U \mapsto \begin{cases} * & \text{if } X \neq U \\ Z & \text{if } X = U \end{cases}.$$

We let $\Delta : tr_Z(U) \to U^Z$ be the map of U–cubes determined by the diagonal map from Z to $\prod_{x\in U} Z$. We define

$$\eta_U = \text{hcofib } F(\Delta) : F(Z) \cong \text{hcofib } F(tr_Z(U)) \longrightarrow \text{hcofib } F(U^Z) = \widehat{cr}^U F.$$

Definition (1.5): For n a natural number, we let $\widehat{cr}^n F$ be the multi-functor

$$\mathcal{S}^{\times n} \xrightarrow{\widehat{cr}^n F} \mathcal{S}'$$

where $\widehat{cr}^n F(M_1, \ldots, M_n)$ is defined by $\widehat{cr}^f F$ with $f : \mathbf{n} \to \mathcal{S}'$ such that $f(i) = M_i$. We observe that for all $\sigma \in \Sigma_n = Hom_{\mathcal{M}_n}(\mathbf{n}, \mathbf{n})$,

$$\widehat{cr}^n F(M_1, \ldots, M_n) \cong \widehat{cr}^n F(M_{\sigma(1)}, \ldots, M_{\sigma(n)})$$

and that

$$\widehat{cr}^n F(M_1, \ldots, M_n) \simeq \widehat{cr}^2[\widehat{cr}^{n-1} F(M_1, \ldots, M_{n-2}, \star)](M_{n-1}, M_n).$$

Lemma (1.6): If F is stably n-excisive and satisfies $E_n(c, \kappa)$, then $\widehat{cr}^n F(X_1, \ldots, X_{n-1}, \ )$ satisfies $E_{n-1}(c, \kappa)$ when $X_1, \ldots, X_{n-1}$ are at least κ-connected.

Proof: By the observations in 1.5, it suffices to show that if F satisfies $E_n(c, \kappa)$ then $\widehat{cr}^2 F(\ , Y)$ satisfies $E_{n-1}(c, \kappa)$ for Y at least κ-connected. Let $\mathcal{X}$ be a strongly $|S| = n$–Cartesian cube such that for all $s \in \mathcal{P}(S)$ the map $\mathcal{X}(\emptyset) \to \mathcal{X}(s)$ is k_s–connected and $k_s \geq \kappa$. We want to show that $\widehat{cr}^2 F(\ , Y)$ applied to $\mathcal{X}$ is $(-c + \Sigma k_s)$-Cartesian or equivalently that the total fiber of the diagram

$$\begin{array}{ccc} F(*) & \longrightarrow & F(\mathcal{X}) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(\mathcal{X} \coprod Y) \end{array}$$

(where we are thinking of Y and $*$ as the constant n–cubical diagrams consisting of identity maps) is $(-c + \Sigma k_s - 1)$-connected. Now $\mathcal{X} \to \mathcal{X} \coprod Y$ is a strongly Cartesian $n+1$ cube such that the maps from the initial object are all at least κ connected and so F satisfying $E_n(c, \kappa)$ implies that F of this cube is $(-c+\Sigma k_s)$-Cartesian. Since F of $* \to Y$ is Cartesian the result follows.

2. The definition of $L_n F$

Once again, let $\mathcal{S}$ and $\mathcal{S}'$ be any of the categories $\mathcal{S}p_*$, $\mathcal{S}pec_b$ or $\mathcal{S}pec$.

Definition (2.1): Let X be a compact pointed space. We define $L_n(X)$ to be the endofunctor of $Func_0(\mathcal{S}, \mathcal{S}')$ defined by

$$\begin{aligned} L_n(X)(F) &= Hom_{\mathcal{M}_n}(\widehat{cr}^* Id(X), \widehat{cr}^* F(X \wedge -)) \\ &= Hom_{\mathcal{M}_n}(X^{\wedge *}, \widehat{cr}^* F(X \wedge \ )). \end{aligned}$$

We let $l_n(S^0)$ be the natural transformation from Id to $L_n(S^0)$ determined by the composite of

$$F \cong lim_{\mathcal{M}_n} tr(F) \xrightarrow{\eta} lim_{\mathcal{M}_n} \widehat{cr}^*(F) \cong Hom_{\mathcal{M}_n}((S^0)^{\wedge *}, \widehat{cr}^* F(S^0 \wedge -)).$$

By restriction to subcategories, we obtain natural transformations

$$r_n(X) : L_n(X) \longrightarrow L_{n-1}(X).$$

By A.4, the $r_n(X)$ are fibrations with fiber

$$\text{fib } (r_n(X)) = Hom_{\Sigma_n}\left(\frac{X^{\wedge n}}{\Delta X^{\wedge n}}, \widehat{cr}^n F(X \wedge -)\right)$$

where Σ_n is identified with $Hom_{\mathcal{M}_n}(\mathbf{n}, \mathbf{n})$ and $\Delta X^{\wedge n}$ is the *pointed fat diagonal.* That is,

$$\Delta X^{\wedge n} = \{(x_1 \wedge \cdots \wedge x_n) \in X^{\wedge n} | \; x_i = x_j \text{ for some } i \neq j\}.$$

We also note that $r_n(S^0) \circ l_n(0) = l_{n-1}(0)$ and *if F is a homotopy functor then $L_n(X)(F)$ is a homotopy functor* by A.1.

For Y a pointed space, we let η_Y be the natural transformation from $Y \wedge F(-)$ to $F(Y \wedge -)$ obtained by the image of the identity via the natural composite:

$$\begin{aligned} Hom_{\mathcal{S}}(Y \wedge X, Y \wedge X) &\cong Hom_{Sp_*}(Y, Hom_{\mathcal{S}}(X, Y \wedge X)) \\ &\xrightarrow{F} Hom_{Sp_*}(Y, Map_{\mathcal{S}'}(F(X), F(Y \wedge X)) \\ &\cong Hom_{\mathcal{S}'}(Y \wedge F(X), F(Y \wedge X)). \end{aligned}$$

We abuse notation and also write η_Y for the induced natural transformation from $id^* \wedge \hat{c}r^* F$ to $\hat{c}r^* F(id \wedge \;)$. We let σ_Y be the natural transformation from $L_n(X)$ to $L_n(Y \wedge X)$ defined by the natural composite:

$$\begin{aligned} L_n(X)(F) =& Hom_{\mathcal{M}_n}(X^{\wedge *}, \widehat{cr}^* F(X \wedge -)) \\ \xrightarrow{id_Y \wedge -}& Hom_{\mathcal{M}_n}(X^{\wedge *} \wedge Y^{\wedge *}, Y^{\wedge *} \wedge \widehat{cr}^* F(X \wedge -)) \\ \cong& \; Hom_{\mathcal{M}_n}((Y \wedge X)^{\wedge *}, Y^{\wedge *} \wedge \widehat{cr}^* F(X \wedge -)) \\ \xrightarrow{\eta_Y}& Hom_{\mathcal{M}_n}((Y \wedge X)^{\wedge *}, \widehat{cr}^* F(Y \wedge X \wedge -)) \\ =& L_n(Y \wedge X)(F) \end{aligned}$$

Definition (2.2): For F a reduced functor from $\mathcal{S}$ to $\mathcal{S}'$ we define

$$L_n F = \text{hocolim}_{\text{k}} \; L_n(S^k) F$$

with structure maps given by σ_{S^1}. We define $l_n : F \to L_nF$ to be the natural transformation determined by $l_n(S^0)$ and taking limits and thus we obtain an inverse limit tower

$$\begin{array}{ccccccc} & & & F & & & \\ \cdots & l_{n+1} & \swarrow & \downarrow{\scriptstyle l_n} & \searrow & l_{n-1} & \cdots \\ \cdots & L_{n+1}F & \xrightarrow{r_{n+1}} & L_nF & \xrightarrow{r_n} & L_{n-1}F & \cdots \end{array}$$

We let $R_nF = \text{hfib } (r_n)$.

Observation (2.3): If $\mathcal{S}'$ is the category $\mathcal{S}pec$ (of spectra) then

$$R_nF = \text{hfib } r_n = \text{hocolim}_k Hom_{\Sigma_n}\left(\frac{(S^k)^{\wedge n}}{\Delta(S^k)^{\wedge n}}, \widehat{cr}^n F(S^k \wedge -)\right)$$

since a fibration of spectra is also a cofibration of spectra and homotopy colimits commute. Similarly, we see by observation A.5 that the following natural diagram is Cartesian (a homotopy pull-back):

$$\begin{array}{ccc} L_n(F) & \xrightarrow{res} & \text{hocolim}_k Hom_{\Sigma_n}\left((S^k)^{\wedge n}, \widehat{cr}^n F(S^k \wedge -)\right) \\ \downarrow{\scriptstyle r_n} & & \downarrow{\scriptstyle i^*} \\ L_{n-1}(F) & \xrightarrow{\alpha} & \text{hocolim}_k Hom_{\Sigma_n}\left(\Delta(S^k)^{\wedge n}, \widehat{cr}^n F(S^k \wedge -)\right). \end{array}$$

PROPOSITION (2.4): If $\mathcal{S}'$ is the category $\mathcal{S}pec_b$ and F satisfies stable n-excision then R_nF is naturally equivalent to

$$\left[\text{hocolim}_\text{k}\ Hom\left((S^k)^{\wedge n}, \widehat{cr}^n F(S^k \wedge -)\right)\right]_{h\Sigma_n}.$$

Proof: This is primarily an application of the Tate map (see appendix B). We first note that since $\frac{(S^k)^{\wedge n}}{\Delta(S^k)^{\wedge n}}$ is a free Σ_n–space, the natural map

$$(1) \qquad R_nF \longrightarrow \text{hocolim}_k \left[Hom\left(\frac{(S^k)^{\wedge n}}{\Delta(S^k)^{\wedge n}}, \widehat{cr}^n F(S^k \wedge -)\right)\right]^{h\Sigma_n}$$

from fixed points to homotopy fixed points is an equivalence. By the Tate map (see B.3), the right hand side of (1) is equivalent to

$$\text{hocolim}_k \left[Hom\left(\frac{(S^k)^{\wedge n}}{\Delta(S^k)^{\wedge n}}, \widehat{cr}^n F(S^k \wedge -)\right)\right]_{h\Sigma_n}.$$

Since F satisfies stable n-excision, the connectivity of $\widehat{cr}^n F(S^k \wedge X)$ eventually goes to infinity linearly with respect to nk. The dimension of $(S^k)^{\wedge n}$ is kn and that of $\Delta(S^k)^{\wedge n}$

is $k(n-1)$. Since $[\ \ ]_{h\Sigma_n}$ preserves connectivity, we see that the right hand side of (1) is equivalent to

$$\text{hocolim}_k \left[Hom\left((S^k)^{\wedge n}, \widehat{cr}^n F(S^k \wedge -)\right)\right]_{h\Sigma_n}$$

and the result now follows from the fact that homotopy colimits commute.

3. Multi-linear functors

We will call a functor F from $\mathcal{S}pec$ to $\mathcal{S}pec$ *additive* if it is reduced and for all $X, Y \in \mathcal{S}pec$:

$$F(X \times Y) \xrightarrow{\simeq} F(X) \times F(Y)$$

(where the map is given by the natural projections). We call a functor G from $\mathcal{S}pec^{\times n}$ to $\mathcal{S}pec$ *n–multi-additive* if for all $X_1, \dots, X_n \in \mathcal{S}pec^{\times n}$, the n–different functors

$$G_{\hat{i}} = G(X_1, \dots, X_{i-1}, -, X_{i+1}, \dots, X_n)$$

are all additive.

Lemma (3.1): Let $G : \mathcal{S}pec^{\times n} \to \mathcal{S}pec$ be multi–additive and $F = G \circ \Delta$ the reduced functor obtained by composition with the diagonal. Then

$$\widehat{cr}^* F \xrightarrow{\simeq} Hom(Hom_{\mathcal{M}_n}(\mathbf{n}, -)_+, F) = \prod_{Hom_{\mathcal{M}_n}(\mathbf{n},-)} F$$

as $\mathcal{M}_n$–diagrams.

Proof: We first observe that for any $X_1, \dots, X_t \in \mathcal{S}pec$,

$$\begin{aligned} F(X_1 \times \cdots \times X_t) =& G(\prod_{i=1}^t X_i, \dots, \prod_{i=1}^t X_i) \\ &\xrightarrow{\simeq} \prod_{\alpha \in Hom_{\text{Sets}}(\mathbf{n},\mathbf{t})} G(X_{\alpha(1)}, \dots, X_{\alpha(n)}). \end{aligned}$$

Thus, we see that for any $U \in \mathcal{M}$ and function $f : U \to \mathcal{S}pec$ we obtain an equivalence of U–cubes $F(U^f) \xrightarrow{\simeq} G^f$ where G^f is given by

$$X \subseteq U \ \mapsto \prod_{\alpha \in Hom_{\text{Sets}}(\mathbf{n},\mathbf{t})} G(f(\alpha(1)), \dots, f(\alpha(n))).$$

Taking homotopy cofibers we obtain a natural transformation (which is an equivalence) η^f :

$$\widehat{cr}^f F \xrightarrow{\eta^f} \prod_{Hom_{\mathcal{M}_n}(\mathbf{n},U)} G(f(\alpha(1), \dots, f(\alpha(n))).$$

It is straightforward to check that η is a natural equivalence and hence the result.

Recall that a functor $F \in Func_0(\mathcal{S}pec, \mathcal{S}pec)$ is *linear* or *excisive* if every co-Cartesian square is taken by F to a Cartesian square. We call a functor G from $\mathcal{S}pec^{\times n}$ to $\mathcal{S}pec$ *n–multi-linear* if for all $X_1, \dots, X_n \in \mathcal{S}^{\times n}$, the n–different functors

$$G_{\hat{i}} = G(X_1, \dots, X_{i-1}, -, X_{i+1}, \dots, X_n)$$

are all linear.

Note: Every linear functor is additive but not every additive functor need be linear. Consider, for example, $\mathbf{H}\pi_0$ from $\mathcal{S}pec$ to Eilenberg-Mac Lane spectra.

THEOREM (3.2): If $G : \mathcal{S}pec_b^{\times n} \to \mathcal{S}pec_b$ is n–multi-linear, then $F = G \circ \Delta$ is such that

i) $L_k F \simeq \begin{cases} * & \text{if } k < n \\ F & \text{if } k \geq n \end{cases}$

ii) $l_k : F \xrightarrow{\simeq} L_k F$ if $k \geq n$.

Proof: We first note that since G lands in bounded below spectra, the natural equivalence obtained by G being multi-linear

$$(S^t)^{\wedge n} \wedge G(X_1, \dots, X_n) \xrightarrow{\simeq} G(S^t X_1, \dots, S^t X_n)$$

implies that $F(S^t X)$ is $(nt - c_X)$–connected for some c_X (depending on X) and all t. If $k < n$ then $Hom\left((S^t)^{\wedge k}, \widehat{cr}^k F(S^t X)\right)$ is at least $((n-k)t - c_X)$–connected. Taking homotopy orbits of this (which preserves connectivity) and the limit with respect to t we see by proposition 1.7 that $R_k F$ is equivalent to $*$ if $k < n$. By lemma 3.1, we also see that $R_k F$ is equivalent to $*$ if $k > n$. Thus, we will be finished once we show l_n is an equivalence and this follows from the fact that l_n is equivalent to the natural composite:

$$\begin{array}{lccc}
(G \text{ is multilinear}) & G & \xrightarrow{\simeq} & \mathrm{hocolim_k}\ Hom((S^k)^{\wedge n}, G(S^k\wedge, \dots, S^k\wedge)) \\
& \| & & \| \\
& F & \xrightarrow{\simeq} & \mathrm{hocolim_k}\ Hom((S^k)^{\wedge n}, F(S^k\wedge)) \\
(\text{adjunction}) & & & \downarrow \cong \\
& & & \mathrm{hocolim_k}\ Hom_{\Sigma_n}((\Sigma_n)_+ \wedge (S^k)^{\wedge n}, F(S^k\wedge)) \\
(\text{adjunction}) & & & \downarrow \cong \\
& & & \mathrm{hocolim_k}\ Hom_{\Sigma_n}((S^k)^{\wedge n}, Hom[\Sigma_n, F(S^k\wedge)]) \\
(\text{lemma 3.1}) & & & \downarrow \simeq \\
& & & \mathrm{hocolim_k}\ Hom((S^k)^{\wedge n}, \hat{cr}^n F(S^k\wedge)) \\
(\text{proposition 2.4.}) & & & \downarrow \simeq \\
& & & R_n F \\
(L_{n-1} F \simeq *) & & & \downarrow \simeq \\
& & & L_n F.
\end{array}$$

4. n–excisive functors

Let $F \in Hom_0(\mathcal{S}pec, \mathcal{S}pec)$. We now recall some terminology and results from [Cal2]. A *strongly co-Cartesian* cube of spectra is a cube of spectra such that every subface is co-Cartesian. A functor F is *n-excisive* is for every strongly co-Cartesian (n+1)–cubical diagram, F of the diagram is Cartesian. By proposition 3.2 of [Cal2], $(n-1)$–excisive implies n–excisive. By proposition 3.4 of [Cal2], the diagonal of a multi-linear functor is n–excisive. The following two lemmas are in [Cal3], but as they are not yet in print we include their proofs here.

LEMMA (4.1): If F is n–excisive then $\widehat{cr}^n F$ is n–multilinear.

Proof: We first note that since

$$\widehat{cr}^k F(X_1, \ldots, X_k) \simeq \widehat{cr}^2[\widehat{cr}^{k-1} F(X_1, \ldots, X_{k-2}, \ )](X_{k-1}, X_k)$$

it suffices to show that if F is n–excisive then $\widehat{cr}^2 F$ is $(n-1)$–excisive in each variable separately. Let $\mathcal{X}$ be a strongly n–coCartesian cube and Y a fixed spectrum. We want to show that $\widehat{cr}^2 F(\ , Y)$ applied to $\mathcal{X}$ is Cartesian or equivalently that the total fiber of the diagram

$$\begin{array}{ccc} F(\mathcal{X} \vee Y) & \longrightarrow & F(\mathcal{X}) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F(*) \end{array}$$

(where we are thinking of Y and $*$ as the constant n–cubical diagrams consisting of identity maps) is equivalent to $*$. Now $\mathcal{X} \vee Y \to \mathcal{X}$ is a strongly coCartesian $n+1$ cube and so F being n–excisive implies that F of this cube is Cartesian but F of $Y \to *$ is also Cartesian and the result follows.

LEMMA (4.2): If F is n–excisive and $\widehat{cr}^n F \simeq *$ then F is $(n-1)$–excisive.

Proof: Let $\mathcal{X}$ be a strongly coCartesian n–cube. Let $\mathcal{Y}$ be the strongly $(n+1)$–cartesian cube obtained by taking the push-out of $\mathcal{X}(\emptyset) \rightarrowtail C\mathcal{X}(\emptyset)$. Since F is n–excisive, F of $\mathcal{Y}$ is Cartesian. By the homotopy invariance of F, the total fiber of F applied to the new face of Y opposite X is equivalent to $\widehat{cr}^n F(Z_1, \ldots, Z_n)$ (where Z_i is equivalent to the homotopy cofiber of $\mathcal{X}(\emptyset) \to \mathcal{X}(i)$) which is equivalent to $*$ by assumption. Thus, the total fiber of F applied to $\mathcal{X}$ is also equivalent to $*$ and F is $(n-1)$–excisive.

What we wish to do now is to use lemma 4.2 to perform induction by degree to functors. For example, using the natural transformation η of 1.5 we obtain a natural transformation from F to $\widehat{cr}^n F \circ \Delta$. Using the natural $\Sigma_n = Hom_{\mathcal{M}_n}(\mathbf{n}, \mathbf{n})$ action on $\widehat{cr}^n F \circ \Delta$, we see that η lands in the Σ_n fixed points and hence we obtain a composite natural transformation

$$F \longrightarrow (\widehat{cr}^n F)^{h\Sigma_n}.$$

This map is always an equivalence on $\widehat{cr}^n$ if F is of degree n. The trouble with this approach is that we don't know if $(\widehat{cr}^n F)^{h\Sigma_n}$ is still stably excisive. What we do, then, is to "dualize" this so we can use homotopy orbits instead.

For U a finite, non-empty set and $f : U \to \mathcal{S}pec$ a function, we define U_f to be the following U-cube of $\mathcal{S}pec$:

$$X \subseteq U \mapsto \bigvee_{u \in U \setminus X} f(u)$$

$$Y \subseteq X \mapsto \bigvee_{v \in U \setminus Y} f(y) \xrightarrow{p} \bigvee_{u \in U \setminus X} f(x)$$

is given by projection. For F a reduced homotopy functor, we define

$$cr_f F = \text{hofib } F(U_f).$$

We obtain a commutative diagram

$$\begin{array}{ccc} cr_f(F) & \longrightarrow & F(\bigvee_{u \in U} f(u)) \\ \big\downarrow \simeq & & \big\downarrow \simeq \\ \widehat{cr}^f(F) & \longleftarrow & F(\prod_{u \in U} f(u)) \end{array}$$

of Σ_n–equivariant maps (and the vertical maps are easily seen to be weak equivalences). Using the sum map we obtain a natural transformation

$$cr_n F(X) \longrightarrow F(\bigvee_{i=1}^{n} X) \xrightarrow{F(+)} F(X)$$

which is Σ_n equivariant (constant action on $F(X)$) and hence a natural composite map:

$$\beta := cr_n F(X)_{h\Sigma_n} \longrightarrow cr_n F(X)_{\Sigma_n} \longrightarrow F(X).$$

Lemma (4.3): If $F \in Func_0(\mathcal{S}pec_b, \mathcal{S}pec_b)$ is n–excisive then $hfib(\beta)$ is $(n-1)$–excisive.

Proof: By lemma 4.1, $cr_n F$ is n–multilinear. Taking the diagonal is thus n–excisive by 3.4 of [Cal2]. Since homotopy colimits of bounded below spectra commute with finite homotopy inverse limits of bounded below spectra, we see that $(cr_n F)_{h\Sigma_n}$ is also n–excisive. Thus, $hfib(\beta)$ is n–excisive. Since taking cross effects preserves homotopy fibrations of functors, it suffices by lemma 4.2 to show that $cr_n \beta$ is an equivalence.

To compute $cr_n (cr_n F)_{h\Sigma_n}$, we recall that since $cr_n F$ is n–multiadditive (3.2):

(4.3.1) $$cr_n F(\bigvee_{i=1}^{t} X_i, \ldots, \bigvee_{i=1}^{t} X_i) \simeq \bigvee_{\alpha \in Hom_{\mathcal{S}ets}(\mathbf{n}, \mathbf{t})} cr_n F(X_{\alpha(1)}, \ldots, X_{\alpha(n)}).$$

This is a Σ_n–equivariant equivalence (weakly) with the Σ_n action on the right-hand side given by $\sigma * (\alpha, v) = (\alpha \circ \sigma^{-1}, \sigma * v)$. The Σ_t action obtained by permuting the X_i–factors is given by $\gamma * (\alpha, v) = (\gamma \circ \alpha, v)$. Thus,

$$cr_n(cr_nF) \simeq \bigvee_{\alpha \in Hom_{\mathcal{M}}(\mathbf{n},\mathbf{n})} cr_nF \cong \Sigma_{n+} \wedge cr_nF.$$

With this identification, the Σ_n action is given by $\sigma * (\alpha \vee x) = (\alpha\sigma^{-1} \vee \sigma * x)$. The map $F(\bigvee X) \stackrel{F(+)}{\longrightarrow} F(X)$ on the n-th cross effects (using identification 4.3.1) is the map taking $\alpha \vee x$ to $\alpha * x$. Since homotopy orbits of spectra commute with finite sums, we see that

$$cr_n(cr_nF_{h\Sigma_n}) \simeq ((\Sigma_n)_+ \wedge cr_nF)_{h\Sigma_n}$$

and $cr_n(\beta)$ is simply the natural equivalence $(G_+ \wedge X)_{hG} \simeq X$ for X a spectrum with G-action.

Theorem (4.4): An element $F \in Func_0^{st(n)}(\mathcal{S}pec_b, \mathcal{S}pec_b)$ is n–excisive if and only if $l_k : F \stackrel{\simeq}{\longrightarrow} L_kF$ for all $k \geq n$.

Proof: We will induct on n. If F is 1–excisive, then the result follows from lemma 3.2. We now assume the result through $n-1$. Since L_n preserves homotopy fibrations we can consider the natural diagram of homotopy fibrations for $k \geq n$

$$\begin{array}{ccccc} \text{hfib}(\beta) & \longrightarrow & cr_n(F)_{h\Sigma_n} & \stackrel{\beta}{\longrightarrow} & F \\ \downarrow{\scriptstyle l_k} & & \downarrow{\scriptstyle l_k} & & \downarrow{\scriptstyle l_k} \\ L_k\text{hfib}(\beta) & \longrightarrow & L_k(cr_n(F)_{h\Sigma_n}) & \stackrel{\beta}{\longrightarrow} & L_k(F). \end{array}$$

The vertical map on the left is an equivalence by lemma 4.3 and induction. The vertical map in the center is an equivalence by lemma 3.2 and lemma 4.1 and hence the vertical map on the right is an equivalence as well.

Conversely, if F is stably n-excisive, then $\widehat{cr}^nF$ is stably excisive in each variable separately. Thus, for each n, $\text{hocolim}_k Hom((S^k)^{\wedge n}, \widehat{cr}^nF(S^k \wedge\ ))$ is n–multi-linear and hence R_nF is n–excisive. Thus, by induction on n, L_nF is n–excisive also and the converse is proved.

Corollary (4.5): The natural transformation l_n is universal for n–excisive functors.

Corollary (4.6): The pair (L_n, l_n) is naturally equivalent to the pair (P_n, p_n) of [Cal3] for $Func_0^{st(n)}(\mathcal{S}pec_b, \mathcal{S}pec_b)$.

Proof: We consider the natural commuting diagram

$$\begin{array}{ccc} F & \xrightarrow{l_n} & L_nF \\ \Big\downarrow{\scriptstyle p_n} & & \Big\downarrow{\scriptstyle p_n} \\ P_nF & \xrightarrow{l_n} & L_nP_nF \end{array}$$

The lower horizontal map is an equivalence by theorem 4.4 since P_nF is n–excisive. The natural transformation $p_n : F \to P_nF$ has the property that for $k \leq n$, the induced map of multi-linearizations of the k-th cross effects are equivalences ([Cal3]) and hence $R_k(p_n)$ is an equivalence for all $k \leq n$. Thus, by induction, $L_k(p_n)$ is an equivalence for all $k \leq n$ and in particular the right hand vertical map is an equivalence.

5. Preliminaries: Modules of Functors with Smash Products

What the homotopy type of a smash product of spectra should be has long been understood and there are several different models for these which are equivalent as spectra. However, the iteration of these constructions is generally only associative up to homotopy. For simplicial constructions this is not suitable since it is not sufficient to have the simplicial identities only up to homotopy. The solution to this problem that we will follow is by M. Bökstedt ([Bö]). His construction involves the notion of a functor with smash product (FSP).

Let $\mathcal{S}_*$ be the category of pointed simplicial sets.

Definition (5.1): A *functor with stabilization* is a functor A from $\mathcal{S}_*$ to $\mathcal{S}_*$ together with a natural transformation

$$\lambda_{X,Y} : X \wedge A(Y) \longrightarrow A(X \wedge Y)$$

such that

i) $\lambda_{X,Y\wedge Z} \circ (id_X \wedge \lambda_{Y,Z}) = \lambda_{X\wedge Y,Z}$ and $\rho_{X\wedge Y,Z} \circ (\lambda_{X,Y} \wedge id_Z) = \lambda_{X,Y\wedge Z} \circ (id_X \wedge \rho_Y, Z)$ where $\rho_{X,Y} : A(X) \wedge Y \longrightarrow A(X \wedge Y)$ is the composite $A(T_{Y,X}) \circ \lambda_{Y,X} \circ T_{A(X),Y}$ ($T =$ twist of two factors).

ii) If X is n–connected, then $A(X)$ is n–connected.

iii) Let $\sigma_X : A(X) \longrightarrow \Omega A(\Sigma X)$ be the adjoint to $\lambda_{S^1,X}$. Then the following limit system stabilizes for each n:

$$\pi_n A(X) \xrightarrow{\sigma_X} \pi_n \Omega A(\Sigma X) \xrightarrow{\sigma_{\Sigma X}} \pi_n \Omega^2 A(\Sigma^2 X) \longrightarrow \cdots$$

For A a functor with stabilization, we let $\mathbf{A}$ be the spectrum with $\mathbf{A}(m) = A(S^m)$ and structure maps given by σ_{S^m} of (iii). We call $\mathbf{A}$ the *spectrum associated to* A. We let $\pi_i(A) = \pi_i \mathbf{A} = \lim_{n\to\infty} \pi_i \Omega^n A(S^n)$.

DEFINITION (5.2): A *functor with smash product* (or just FSP) is a functor with stabilization, A, together with natural transformations:

$$\begin{aligned} 1_X &: X \longrightarrow A(X) \\ \mu_{X,Y} &: A(X) \wedge A(Y) \longrightarrow A(X \wedge Y) \end{aligned}$$

such that

$$\begin{aligned} \mu(\mu \wedge id) &= \mu(id \wedge \mu) \\ \mu(1_X \wedge 1_Y) &= 1_{X \wedge Y} \\ \lambda_{X,Y} &= \mu_{X,Y}(1_X \wedge id_{A(Y)}) \\ \rho_{X,Y} &= \mu_{X,Y} \circ (id_{A(X)} \wedge 1_Y) \end{aligned}$$

For A and A' FSPs, a *morphism* η from A to A' of FSPs is a morphism of functors with stabilizations $A \xrightarrow{\eta} A'$ which strictly commutes with the natural transformations μ and μ'. We will also insist that our morphisms of FSPs strictly preserve the units. That is, our morphisms of FSPs are assumed to be *unital.*

Example: Let A be a ring. We define the FSP $\underline{A}$ by

$$X \mapsto A \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X]$$

where $\tilde{\mathbf{Z}}[X] = \mathbf{Z}[X]/\mathbf{Z}[*]$. The multiplication is given by sending smash to tensor followed by multiplication in A:

$$\begin{aligned} A(X) \wedge A(Y) &\to (A \otimes_{\mathbf{Z}} A) \otimes_{\mathbf{Z}} \tilde{\mathbf{Z}}[X \wedge Y] \\ &\to A(X \wedge Y) \end{aligned}$$

and the unit is given by the inclusion

$$X \longrightarrow A(X).$$

$$x \mapsto id_A \otimes 1 \cdot x$$

DEFINITION (5.3): Let A be an FSP and T a functor with stabilization. A *structure of left A–module* on T is a natural transformation

$$l_{X,Y} : A(X) \wedge T(Y) \longrightarrow T(X \wedge Y)$$

such that

$$\begin{aligned} l(\mu \wedge id) &= l(id \wedge l) \\ \lambda_{X,Y} &= l_{X,Y}(1_X \wedge id_{T(Y)}) \end{aligned}$$

The notion of right A–module is defined similarly.

Example: If we write id for the identity FSP (sending X to X), then the category of functors with stabilization is isomorphic to the category of right (left) id–modules.

FIRST PROPERTIES OF A-MOD (5.4): Let A be an FSP. We will write $A - Mod$ for the category of left A–modules. We consider this as a continuous category by topologizing $Hom_A(M, N)$ with the subspace topology of $\prod_{i=1}^{\infty} Hom(M(S^i), N(S^i))$ (using the evaluation of the modules only on spheres). A map of A–modules is an *equivalence* if the map of associated spectra is a weak equivalence of spectra. We say a set of A-modules $\{M_\alpha\}_{\alpha\in\{\mathcal{I}\}}$ is *globally stable* if for each space X and number n, there is $N(X, n)$ such that $\pi_n(\sigma_{\Sigma^{N(X,n)}X})$ is an equivalence for all M_α. That is, for a given X and n, there is a point at which condition (iii) is satisfied for all M_α simultaneously. The following statements are easily verified and whose details are left to the interested reader.

(5.4.1): The category of A–modules is closed under finite sum and finite products (defined pointwise). We note that $A-Mod$ is not in general closed under arbitrary sums or products as condition (iii) may fail. However, given an arbitrary set of A–modules which is globally stable, then its coproduct is once again an A–module.

(5.4.2): The category of A–modules and simplicial realizations. If M_* is a simplicial A–module which is globally stable then $|M_*|$ is again naturally an A–module. Note, we will always be using the fat realization which is obtained by using only the face relations.

(5.4.3): The category of A–modules and homotopy colimits. Let $\mathcal{I}$ be an indexing category and M_* an $\mathcal{I}$ diagram of A–modules which is globally stable. Then $\mathrm{hocolim}_{\mathcal{I}} M_*$ is again naturally an A–module by using 5.4.1 and 5.4.2. In particular, the category of A-modules is closed under homotopy colimits of finite diagrams.

(5.4.4): Given a compact pointed space X and an A-module M, we define $X \wedge M$ and $Hom(X, M)$ to be the A–modules

$$(X \wedge M)(Y) = X \wedge M(Y) \quad \text{and} \quad Hom(X, M)(Y) = Hom(X, M(Y))$$

(with the obvious structure maps). As for spectra, we have natural homeomorphisms for pointed compact space X, Y and A-modules M and N:

$$Hom(X \wedge Y, M) \cong Hom(X, Hom(Y, M))$$

$$Hom_A(X \wedge M, N) \cong Hom_A(X, Hom(M, N))$$

(the first is as A-modules, the second is as spaces). We also define, for $n \geq 0$, $M(n)$ to be the new A-module defined by $M(n)(Y) = M(S^n \wedge Y)$. By condition (iii), the map of A–modules $S^n \wedge M \to M(n)$ is an equivalence of A–modules.

(5.4.5): $\Omega - A$-modules: We will call an A–module an $\Omega - A$-module if its associated spectrum is an Ω-spectrum. Every A-module M is naturally equivalent to an $\Omega - A$-module by

$$\tilde{M} = \mathrm{hocolim}_n \; Hom(S^n, M(n))$$

(which is again an A-module by 5.4.3 and 5.4.4). However, we want to have a little more structure than this construction gives us. Let I be the category of finite non-empty sets and inclusions as morphisms. For $x \in I$, we let S^x be the sphere indexed on x. For $\alpha \in Hom_I(x,y)$, we let $\alpha_* : Map(S^x, M(S^x)) \to Map(S^y, M(S^y))$ be defined by suspensions followed by λ and then reordering (see for example [P-W] or [D-M]). By a result of Bökstedt, [Bö],

$$\mathrm{hocolim}_n\ Hom(S^n, M(n)) \xrightarrow{\simeq} \mathrm{hocolim}_{x\in I}\ Hom(S^x, M(S^x))$$

and we let $\Omega^\infty M$ be $\mathrm{hocolim}_{x\in I}\ Hom(S^x, M(S^x))$. One reason we want to use this model for constructing our infinite loop models is the following. There is a natural A-module equivalence $\Omega^\infty(\Omega^\infty M) \longrightarrow \Omega^\infty M$ defined by the composite

$$\begin{aligned}\Omega^\infty(\Omega^\infty M) &= \mathrm{hocolim}_{x\in I}\mathrm{hocolim}_{y\in I}\mathrm{Map}(S^x, \mathrm{Map}(S^y, M(S^x \wedge S^y)))\\ &\cong \mathrm{hocolim}_{x,y\in I\times I} Map(S^{x\cup y}, M(S^{x\cup y}))\\ &\xrightarrow{\cup} \mathrm{hocolim}_{z\in I} Map(S^z, M(S^z))\\ &= \Omega^\infty M\end{aligned}$$

where the map "$\cup$" is the map of homotopy colimits induced by the functor $I \times I \to I$ obtained by concatenation of sets. Another advantage of this model is that if A is an FSP then $\Omega^\infty A$ is again an FSP with multiplication determined by

$$\Omega^\infty A(X) \wedge \Omega^\infty A(Y)$$

$$\big\downarrow =$$

$$\mathrm{hocolim}_{x\in I}\mathrm{Map}(S^x, A(S^x \wedge X)) \wedge \mathrm{hocolim}_{y\in I}\mathrm{Map}(S^y, A(S^y \wedge Y))$$

$$\big\downarrow \cong$$

$$\mathrm{hocolim}_{x,y\in I\times I}\mathrm{Map}(S^x, A(S^x \wedge X)) \wedge \mathrm{Map}(S^y, A(S^y \wedge Y))$$

$$\big\downarrow a$$

$$\mathrm{hocolim}_{x,y\in I\times I}\mathrm{Map}(S^x \wedge S^y, A(S^x \wedge X) \wedge A(S^y \wedge Y))$$

$$\big\downarrow b$$

$$\mathrm{hocolim}_{x,y\in I\times I}\mathrm{Map}(S^{x\cup y}, A(S^x \wedge X \wedge S^y \wedge Y))$$

$$\big\downarrow \cong$$

$$\mathrm{hocolim}_{x,y\in I\times I}\mathrm{Map}(S^{x\cup y}, A(S^{x\cup y} \wedge X \wedge \wedge Y))$$

$$\big\downarrow \cup$$

$$\mathrm{hocolim}_{z\in I}\mathrm{Map}(S^z, A(S^z \wedge X \wedge Y))$$

where a is the smashing of maps and b is from the FSP multiplication of A. Similarly, if M is a left A-module, we can define a natural left $\Omega^\infty A$–module structure on $\Omega^\infty M$.

Let A and B be FSPs. A *homotopy functor* F from $A - Mod$ to $B - Mod$ is a continuous functor which takes equivalences to equivalences. A functor F is *reduced* if $F(*) = *$. By 5.4, we can talk about k–coCartesian cubes of modules and we define a homotopy functor of modules to be *(stably) n–excisive* if the analogous statement used for functors of $\mathcal{S}pec_b$ holds. We let

$$Func_0^{st(n)}(A - Mod, B - Mod)$$

be the category of stably n-excisive reduced homotopy functors with continuous natural transformations as morphisms.

6. n–excisive functors of modules

Given $F \in Func_0(A - Mod, B - Mod)$, we note that by 5.4.3 and section 1 that we can define $\widehat{cr}^* F$ as we did for functors of spectra and we obtain a functor

$$Func_0(A - Mod, B - Mod) \xrightarrow{\widehat{cr}} Func(\mathcal{M}, Func_0(A - Mod, B - Mod)).$$

For X a pointed space, we can (using 5.4.4), define the endofunctor $L_n(X)$ of $Func_0(A - Mod, B - Mod)$ by

$$L_n(X)(F) = Hom_{\mathcal{M}_n}(X^*, \widehat{cr}^* F(X \wedge\ ))$$

and similarly

$$L_n F = \mathrm{hocolim}_k\ L_n(S^k)F$$

with structure maps given by σ_{S^1}. We once again obtain an inverse limit tower with natural transformations $l_n : F \to L_n F$ and $r_n : L_n F \to L_{n-1}F$. When F satisfies stable n-excision, then $R_n F = \mathrm{hfib}\ (r_n)$ is naturally equivalent to

$$\left[\mathrm{hocolim}_k\ Hom\left((S^k)^{\wedge n}, \widehat{cr}^n F(S^k \wedge -)\right)\right]_{h\Sigma_n}$$

(essentially by proposition 2.4 again). Since homotopy orbits preserve connectivity, we see by induction that L_n is naturally an endofunctor of $Func_0^{st(n)}(A - Mod, B - Mod)$ again.

We define the notion of F being *additive* or *linear* as we did for functors of $\mathcal{S}pec$. It is straightforward to see that the results of section 3 and 4 still hold for functors of module categories and hence we can immediately deduce the following result.

Theorem (6.1): An element $F \in Func_0^{st(n)}(A - Mod, B - Mod)$ is n–excisive if and only if $l_k : F \xrightarrow{\simeq} L_k F$ for all $k \geq n$. The natural transformation l_n is universal for n–excisive functors.

In order to state our next result we need to make some further definitions. We will say that $F \in Func_0(A - Mod, B - Mod)$ satisfies the *limit axiom* if F commutes with filtered colimits. That is, if $\mathcal{I}$ is a filtered limit system and M_* is a globally stable $\mathcal{I}$-diagram in $A - Mod$ then

$$\mathrm{hocolim}_{\mathcal{I}} F(M_*) \xrightarrow{\simeq} F(\mathrm{hocolim}_{\mathcal{I}} M_*).$$

For n a natural number, we set

$$A^n = \mathbf{n}_+ \wedge A = \{1, \ldots, n\}_+ \wedge A.$$

Our objective for the remainder of this section is to prove the following result.

THEOREM (6.2): Let F and G be n–excisive functors in $Func_0(A - Mod, B - Mod)$ which satisfy the limit axiom and let $\eta : F \to G$ be a continuous natural transformation between them. If η_{A^n} is an equivalence then η_M is an equivalence for all $M \in A - Mod$.

Since several steps are involved in establishing this result, we first outline the strategy. We first show that a linear functor which satisfies the limit axiom commutes with realizations of simplicial A–modules (6.3). By induction, it will follow that n–excisive functors which satisfy the limit axiom commute with realizations as well (6.4). By a *free A-module* we will mean an A-module that is equivalent (as A-modules) to $X \wedge A$ for some based space X. Since F and G are n-excisive, if η_{A^n} is an equivalence then η_{A^t} will be an equivalence for all t and hence by the limit axiom that η_M is an equivalence for all free modules M (6.5). Thus, we will be finished once we show that every object of $A - Mod$ has a simplicial resolution by free A–modules (6.6). This will be proved by following the standard method in algebra for obtaining simplicial free resolutions for modules by using the adjoint pair of the free functor and the forgetful functor (to sets).

PROPOSITION (6.3): If $F \in Hom_0(A - Mod, B - Mod)$ is linear and satisfies the limit axiom then F commutes with realizations that are well defined. That is, if M_* is a simplicial A–module which is globally stable such that $F(M_*)$ is globally stable (as B-modules) then

$$\|F(M_*)\| = \mathrm{hocolim}_{\Delta^{op}} F(M_*) \xrightarrow{\simeq} F(\mathrm{hocolim}_{\Delta^{op}} M_*) = F(\|M_*\|).$$

Proof: Let M_* be a globally stable simplicial A-module and for each $n \geq 0$, let $sk_n M_*$ be the simplicial n–skeleton of M_*. Since

$$\|M_*\| = \mathrm{hocolim}_{\Delta^{op}} M_* = \mathrm{hocolim}_n \ \mathrm{hocolim}_{\Delta^{op}} \ sk_n M_*$$

it follows by the facts that homotopy colimits commute, F preserves weak equivalences and has the limit axiom that it suffices to ask if

$$\|F(sk_n M_*)\| \xrightarrow{?} F(\|sk_n M_*\|)$$

is an equivalence for all n. We will induct on n. The case $n = 0$ follows from the homotopy invariance for F. Assume the result is known through $n-1$. We recall that the n–skeleton fits into a co-Cartesian cube:

$$\begin{array}{ccc} M_n \wedge \partial\Delta^n_+ & \longrightarrow & M_n \wedge \Delta^n_+ \\ \big\downarrow & & \big\downarrow \\ \|sk_{n-1}M_*\| & \longrightarrow & \|sk_n M_*\|. \end{array}$$

By naturallity we obtain a strictly commuting cube as follows:

$$\begin{array}{ccc} F(M_n \wedge \partial\Delta^n_+) & \longrightarrow & F(M_n \wedge \Delta^n_+) \\ \big\downarrow & & \big\downarrow \\ F(\|sk_{n-1}M_*\|) & \longrightarrow & F(\|sk_n M_*\|) \end{array}$$

$$\begin{array}{ccc} F(M_n) \wedge \partial\Delta^n_+ & \longrightarrow & F(M_n) \wedge \Delta^n_+ \\ \big\downarrow & & \big\downarrow \\ \|sk_{n-1}F(M_*)\| & \longrightarrow & \|sk_n F(M_*)\| \end{array}$$

The diagonal maps on the left are equivalences by the induction step. The upper diagonal map on the right is an equivalence by the homotopy invariance of F. The front face is co-Cartesian (hence Cartesian, as spectra) by construction and the back face is Cartesian because F is linear–thus the lower diagonal map on the right is also an equivalence and we are done.

Corollary (6.4): If $F \in Hom_0(A - Mod, B - Mod)$ is n–excisive and satisfies the limit axiom then F commutes with realizations that are well defined.

Proof: We will induct on n. The case for $n = 1$ was done as proposition 6.3. Assume the result is true through $n - 1$. Since $L_{n-1}F$ is $n - 1$ excisive and satisfies the limit axiom, it commutes with realizations by the inductive step. By the natural homotopy cofibration sequence

$$R_nF \longrightarrow L_nF \xrightarrow{r_n} L_{n-1}F$$

and the fact that homotopy colimits commute it suffices to show that R_nF commutes with realizations when F is n–excisive. Since homotopy orbits are a colimit construction, it suffices to show that $\widehat{cr}^n F \circ \Delta$ commutes with realizations when F is n–excisive. By the Eilenberg-Zilber theorem, it suffices to show that $\widehat{cr}^n F$ commutes with realizations in each variable separately but this follows from proposition 6.3 again since $\widehat{cr}^n F$ is n–multi-linear when F is n–excisive.

Corollary (6.5): Let F and G be n–excisive functors in $Func_0(A - Mod, B - Mod)$ and $\eta : F \to G$ a continuous natural transformation between them. If η_{A^n} is an equivalence then η_M is an equivalence for all free A–modules M.

Proof: Since F and G are homotopy functors and every pointed space (in our category) is weakly equivalent to the realization of a simplicial set, it suffices to check the result for A–modules of the form $|X_*| \wedge A \simeq \|X_* \wedge A\|$. By corollary (6.4), F and G commute with realizations of this type and so it suffices to show η is an equivalence for free modules of the form $Z_+ \wedge A$ where Z is a set. By the limit axiom, it suffices to show η is an equivalence when Z is finite. By the natural equivalence (Z-finite) $Z_+ \wedge A \xrightarrow{\simeq} \prod_Z A$ it suffices to show η is an equivalence for finite products of A or equivalently that

$$\widehat{cr}^z F(A) \xrightarrow{\widehat{cr}^z \eta(A)} \widehat{cr}^z G(A)$$

is an equivalence for all z. Since F and G are n–excisive, $\widehat{cr}^z F \simeq * \simeq \widehat{cr}^z G$ for all $z > n$. For $k < n$, $A^{\times k}$ is a natural deformation retract of $A^{\times n}$, thus if $\eta_{A^{\times n}}$ is an equivalence then $\eta_{A^{\times k}}$ is an equivalence for all $k \leq n$. In particular, $\eta_{A^{\times n}}$ an equivalence implies $\widehat{cr}^z \eta(A)$ is an equivalence for all $z \leq n$ and hence the result.

Proposition (6.6): Every $M \in A - Mod$ is naturally equivalent (as an A-module) to the realization of a simplicial object of free A–modules.

Proof: Let Ω_0^∞ be the functor from $A - Mod$ to pointed spaces given by (see 5.4.5):

$$\Omega_0^\infty M = \Omega^\infty M(S^0) = \text{hocolim}_{x \in I}\ Hom(S^x, M(S^x)).$$

Let $\Omega^\infty(A \wedge\ )$ be the functor from pointed spaces to $A - Mod$ given by

$$\Omega^\infty(A \wedge\ )(X) = \text{hocolim}_{x \in I}\ Hom(S^x, A(S^x) \wedge X).$$

We let $A[\ ]$ be the endofunctor of $A - Mod$ defined by

$$\begin{aligned} A[M] &= \Omega^\infty(A \wedge\ ) \circ (\Omega_0^\infty)(M) \\ &= \text{hocolim}_{x,y \in I^2}\ Hom(S^x, A(S^x) \wedge Hom(S^y, M(S^y))) \\ &= \Omega^\infty(A \wedge \Omega^\infty(M)(S^0)). \end{aligned}$$

We have a natural transformation ϵ of $A - Mod$ from $A[\ ]$ to $\Omega^\infty M$ induced by the left multiplication map and concatenation of sets. Using the identity in $A(S^0)$, we also have a natural transformation σ from $A[\ ]$ to $A[A[\ ]]$ as functors to $A - Mod$ determined by the composite

$$\begin{aligned} \Omega^\infty(A \wedge\ )(X) &= \text{hocolim}_{x \in I}\ Hom(S^x, A(S^x) \wedge X) \\ &\xrightarrow{\wedge 1_A} \text{hocolim}_{x \in I}\ Hom(S^x \wedge S^0, A(S^x) \wedge A(S^0) \wedge X) \\ &\xrightarrow{inc_*} \text{hocolim}_{x,y \in I^2}\ Hom(S^x \wedge S^y, A(S^x) \wedge A(S^y) \wedge X). \end{aligned}$$

We can form a simplicial object of endofunctors of $A - Mod$ (essentially from a cotriple) by

$$\begin{aligned}[n] \mapsto A[\ ]^{n+1} &= \overbrace{A[\ ] \circ \cdots \circ A[\ ]}^{n+1 \text{ times}} \\ d_i &= A[\ ]^{n-i} \epsilon A[\ ]^i \\ s_j &= A[\ ]^{n-i} \sigma A[\ ]^j.\end{aligned}$$

We obtain a map of simplicial objects $A[\ ]^{*+1} \to \Omega^\infty$ using ϵ if we consider Ω^∞ as the trivial constant simplicial object. We want to show that this map is an equivalence. Using the standard first quadrant spectral sequence for computing a simplicial spectrum–we see that it suffices to show that the map of simplicial abelian groups π_k is an equivalence for all k. Since every spectrum involved is an Ω–spectrum–it thus suffices to show that the map of simplicial *spaces* obtained by evaluation at S^0 is an equivalence. However, at this level, we can construct a simplicial homotopy by using the identity of A in $A(S^0)$ as one does in ordinary algebra.

Proof of theorem (6.2): Since every A–module has a resolution by free A–modules (6.4) and F and G commute with realizations (6.2) and agree on free modules (6.3) then η is an equivalence for all A–modules by the realization lemma.

7. Classification of n–excisive functors by modules

We now wish to classify n–excisive functors of module categories which satisfy the limit axiom. Our classification will be by bi-modules. We first establish some FSPs and modules over them that we wish to use, then we define the tensor product for modules over FSPs and finally we establish our classification result as theorem (7.6).

In this section and the next, we make the following change in notation. Recall (5.4.5) that we let I be the category of finite non-empty sets and inclusion maps. Considering $\mathbf{N}$ as a well ordered subcategory of I we obtain a natural transformation

$$L_n F = \text{hocolim}_{k \in \mathbf{N}} L_n(S^k)F \xrightarrow{\epsilon_n} \text{hocolim}_{x \in I} L_n(S^x)F.$$

LEMMA (7.0): For $F \in Hom_0^{st(n)}(A - Mod, B - Mod)$, ϵ_n is a weak equivalence.

Proof: By Bökstedt, this is a weak equivalence if the connectivity of the maps $L_n F(\alpha)$ for $\alpha \in Hom_I(x, y)$ tend to infinity uniformly with $|x|$. One can consider this question directly but we will use induction. We first observe that since F is stably 1-excisive, $S^x \wedge F(M(S^k)) \longrightarrow F(M(S^x \wedge S^k))$ is $(2|k|) - c$–connected for all $x \in I$ and some fixed $c \geq 0$ depending on M. In particular, $F(M(S^k))$ is at least $[|k| - (c+1)]$-connected. Consider the composite map for $x, k \in I$:

$$Map(S^k, F(M(S^k))) \xrightarrow{S^x \wedge id} Map(S^x \wedge S^k, S^x \wedge F(M(S^k))) \longrightarrow Map(S^{x \cup k}, F(M(S^{x \cup k}))).$$

The first map is (at least) $[2(|k|-(c+1))]$-connected by the Freudenthal suspension theorem and the second is $[2|k|-(c+1)-(|k|+|x|)=|k|-(c+|x|+1)]$-connected. Thus, the connectivity of $L(\alpha)$ for $\alpha \in Hom_I(k,x)$ is at least $k+(c-x)$ which tends to infinity linearly with respect to k. Hence, ϵ_1 is an equivalence for functors which satisfy stable excision. By induction and the fact that homotopy colimits commutes it suffices to show that

$$\chi_n F = \mathrm{hocolim}_{k\in\mathbf{N}} Map((S^k)^{\wedge n}, \widehat{cr}^n F(S^k)) \longrightarrow \mathrm{hocolim}_{x\in I} Map((S^x)^{\wedge n}, \widehat{cr}^n F(S^x))$$

is a weak equivalence. This is the diagonal of

$$\begin{array}{c} \mathrm{hocolim}_{k_1,\ldots,k_n\in\mathbf{N}^{\times n}} Map(S^{k_1},\ldots,S^{k_n}, \widehat{cr}^n F(S^{k_1},\ldots,S^{k_n})) \\ \Big\downarrow \\ \mathrm{hocolim}_{x_1,\ldots,x_n\in I^{\times n}} Map(S^{x_1},\ldots,S^{x_n}, \widehat{cr}^n F(S^{x_1},\ldots,S^{x_n})). \end{array}$$

which is an equivalence since $\widehat{cr}^n F$ is eventually stably 1-excisive in each variable if F is stably excisive. The result follows by finality since the diagonals are final subcategories in $\mathbf{N}^{\times n}$ and $I^{\times n}$.

We will now use the equivalent model with homotopy colimit over I as our definition of $L_n F$.

<u>The ring $L_n S[M_n A]$</u>

Let B be an FSP. Let $B[\ ]$ be the element of $Func_0(A-Mod, B-Mod)$ defined by

$$M \mapsto B \wedge \Omega^\infty M(S^0)$$

The next lemma follows almost immediately from the definitions and left to the reader.

Lemma (7.1): There is a natural equivalence of $\mathcal{M}_n$-functors:

$$\widehat{cr}^* B[\star] \simeq B \wedge (\Omega^\infty \star (S^0))^{\wedge *}.$$

For X a pointed space, we will abuse notation and write

$$L_n B[X] = \mathrm{hocolim}_{x\in I}\ Map_{\mathcal{M}_n}((S^x)^{\wedge *}, B(S^x \wedge\ \ ) \wedge X^{\wedge *})$$

and for M an A–module, we will write

$$L_n B[M] = L_n B[\Omega^\infty M(S^0)]$$

which is naturally equivalent to $L_nB[M]$ as defined in section 6 (though not strictly equal).

Let $\alpha : X \wedge Y \to Z$ be a pointed map. We let α also represent the natural composite map

$$L_nB[X] \wedge L_nB[Y]$$
$$\big\downarrow =$$
$$\mathrm{hocolim}_{x\in I} Map_{\mathcal{M}_n}((S^x)^{\wedge *}, B(S^x) \wedge X^{\wedge *}) \wedge \mathrm{hocolim}_{y\in I} Map_{\mathcal{M}_n}((S^y)^{\wedge *}, B(S^y) \wedge Y^{\wedge *})$$
$$\big\downarrow \wedge$$
$$\mathrm{hocolim}_{x,y\in I^2} Map_{\mathcal{M}_n}((S^x \wedge S^y)^{\wedge *}, B(S^x) \wedge X^{\wedge *} \wedge B(S^y) \wedge Y^{\wedge *})$$
$$\big\downarrow \mathrm{twist}$$
$$\mathrm{hocolim}_{x,y\in I^2} Map_{\mathcal{M}_n}((S^x \wedge S^y)^{\wedge *}, B(S^x) \wedge B(S^y) \wedge (X \wedge Y)^{\wedge *})$$
$$\big\downarrow (\alpha\wedge\mu)_*$$
$$\mathrm{hocolim}_{x,y\in I^2} Map_{\mathcal{M}_n}((S^{x\sqcup y})^{\wedge *}, B(S^{x\sqcup y}) \wedge Z^{\wedge *})$$
$$\big\downarrow \sqcup_*$$
$$\mathrm{hocolim}_{z\in I}((S^z)^{\wedge *}, B(S^z) \wedge Z^{\wedge *})$$
$$\big\downarrow =$$
$$L_nB[Z].$$

We note that if $\mu : X \wedge X \longrightarrow X$ is a strictly associative pairing with identity given by $\epsilon : S^0 \to X$ (that is, (μ, ϵ) give X the structure of a pointed topological monoid) then μ and ϵ give $L_nB[X]$ the structure of an FSP. Additionally, if we have a pointed map $\eta : X \wedge Y \to Y$ which is strictly associative with respect to μ and unital with respect to ϵ (that is, Y is a left X–space), then η naturally gives $L_nB[Y]$ the structure of a left module over $L_nB[X]$. A similar observation applies to produce right module actions.

Given an FSP A, $\Omega^\infty A(S^0)$ is naturally a pointed topological monoid. Similarly, if M is an A–module, then $\Omega^\infty M(S^0)$ is a left $\Omega^\infty A(S^0)$–space. Thus, we have proved the following lemma.

Lemma (7.2): If A and B are FSPs, then $L_nB[A]$ is again naturally an FSP and $L_nB[\ ]$ is a functor from $A - Mod$ to $L_nB[A] - Mod$.

We will write $Hom(\mathbf{n}_+, \mathbf{n}_+ \wedge A)$ as $M_n(A)$ and we note that this is again an FSP whose multiplication is given by composition followed by multiplication (usual matrix multiplication):

$$Hom(\mathbf{n}_+, \mathbf{n}_+ \wedge A(X)) \wedge Hom(\mathbf{n}_+, \mathbf{n}_+ \wedge A(Y))$$

$$\Big\downarrow{\scriptstyle \alpha\wedge\beta\mapsto(\alpha\wedge id_{A(Y)})\circ\beta}$$

(2) $$Hom(\mathbf{n}_+, \mathbf{n}_+ \wedge A(X) \wedge A(Y))$$

$$\Big\downarrow{\scriptstyle \mu_*}$$

$$Hom(\mathbf{n}_+, \mathbf{n}_+ \wedge A(X \wedge Y)).$$

We note that we have a natural linear (1-excisive and reduced) homotopy functor (which satisfies the limit axiom)

$$Hom(\mathbf{n}_+, \ ) : A - Mod \longrightarrow M_n(A) - Mod$$

where the $M_n(A)$-module action is defined similar to that in (2).

Let S be the FSP corresponding to the identity. Thus, $S(X) = X$ and $\mu = id$. We write this FSP as S for the sphere spectrum–which is the associated spectrum of S. Every functor with stabilization is naturally both a left and right S-module. Note that if X is a pointed topological monoid then $S \wedge X$ becomes an FSP with

$$\begin{aligned}(S \wedge X)(Y) \wedge (S \wedge X)(Z) =& S(Y) \wedge X \wedge S(Z) \wedge X \\ &\xrightarrow{\text{twist}} S(Y) \wedge S(Z) \wedge X \wedge X \\ &\xrightarrow{\mu_S \wedge \mu} S(Y \wedge Z) \wedge X = (S \wedge X)(Y \wedge Z).\end{aligned}$$

If F is a functor with stabilization and there is a natural transformation from $F \wedge X$ to F which is strictly associative and unital with respect to the product in X, then F is naturally a right $S \wedge X$–module.

For A and B FSPs, an *A-B-bimodule* M is a functor with stabilization which is a left A-module and a right B-module such that the following commutes:

$$\begin{array}{ccc} A(X) \wedge M(Y) \wedge B(Z) & \xrightarrow{id \wedge r} & A(X) \wedge M(Y \wedge Z) \\ \Big\downarrow{\scriptstyle l \wedge id} & & \Big\downarrow{\scriptstyle l} \\ M(X \wedge Y) \wedge B(Z) & \xrightarrow{r} & M(X \wedge Y \wedge Z). \end{array}$$

Proposition (7.3): If F is a functor from $A - Mod$ to $B - Mod$, then:

(1) $L_n S[M_n A]$ is an FSP

(2) $L_n F(\Omega^\infty(\mathbf{n}_+ \wedge M))$ is naturally a $B - L_n S[M_n A]$–bimodule

(3) $L_n S[Hom(\mathbf{n}_+, \ )]$ is naturally an n–excisive functor from $A{-}Mod$ to $L_n S[M_n A]{-}Mod$.

Proof: Parts (1) and (3) are restatements of corollary 7.2. We first define right action of the pointed topological monoid $\Omega^\infty M_n(A)(S^0)$ on $F(\Omega^\infty(\mathbf{n}_+ \wedge M))$. Consider the natural composite for $x \in I$:

$$\begin{array}{c} Hom(\mathbf{n}_+ \wedge S^x, \mathbf{n}_+ \wedge A(S^x)) \\ \Big\downarrow {\scriptstyle id \wedge M} \\ Hom_A(\mathbf{n}_+ \wedge S^x \wedge M, \mathbf{n}_+ \wedge A(S^x) \wedge M) \\ \Big\downarrow {\scriptstyle \mu_*} \\ Hom_A(\mathbf{n}_+ \wedge S^x \wedge M, \mathbf{n}_+ \wedge M(S^x)) \\ \Big\downarrow {\scriptstyle adj} \\ Hom_A(\mathbf{n}_+ \wedge M,\ Hom(S^x, \mathbf{n}_+ \wedge M(S^x)) \\ \Big\downarrow {\scriptstyle \Omega^\infty} \\ Hom_A(\Omega^\infty(\mathbf{n}_+ \wedge M),\ \Omega^\infty Hom(S^x, \mathbf{n}_+ \wedge M(S^x))). \end{array}$$

Taking the homotopy colimit over $x \in I$ and using the natural A–module map $\Omega^\infty\Omega^\infty \to \Omega^\infty$ (5.4.5) we obtain a natural continuous map of unital monoids (the right-hand side is by composition):

$$\Omega^\infty M_n(A)(S^0) \longrightarrow Hom_A(\Omega^\infty(\mathbf{n}_+ \wedge M), \Omega^\infty(\mathbf{n}_+ \wedge M)).$$

Since F is a continuous functor, we obtain a continuous map to

$$Hom_B(F(\Omega^\infty(\mathbf{n}_+ \wedge M)), F(\Omega^\infty(\mathbf{n}_+ \wedge M))).$$

Using this map and composition of morphisms, we obtain a *right* action of $\Omega^\infty M_n A(S^0)$ on $F(\Omega^\infty(\mathbf{n}_+ \wedge M))$. We now observe that the natural B-module map

$$F(\Omega^\infty(\mathbf{n}_+ \wedge M)) \wedge \Omega^\infty M_n A(S^0) \longrightarrow F(\Omega^\infty(\mathbf{n}_+ \wedge M))$$

we have defined produces a natural transformation of $\mathcal{M}$-B-modules

$$\widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)) \wedge \Omega^\infty M_n A(S^0)^{\wedge *} \longrightarrow \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)).$$

Finally, our right action of $L_n S[M_n(A)]$ is defined by:

$$\begin{array}{c} L_n F(\Omega^\infty(\mathbf{n}_+ \wedge M))(X) \wedge L_n S[M_n(A)](Y) \\ \Big\downarrow = \\ \text{hocolim}_{x\in I} Map_{\mathcal{M}_n}[S^x)^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^x \wedge X))] \end{array}$$

$$\wedge \ \mathrm{hocolim}_{y\in I} Map_{\mathcal{M}_n}[(S^y)^{\wedge *}, S^y \wedge Y \wedge \Omega^\infty M_n(A)(S^0)]$$

$$\big\downarrow \cong$$

$$\mathrm{hocolim}_{x,y\in I\times I} Map_{\mathcal{M}_n}[(S^x)^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^x \wedge X))]$$

$$\wedge \ Map_{\mathcal{M}_n}[(S^y)^{\wedge *}, S^y \wedge Y \wedge \Omega^\infty M_n(A)(S^0)]$$

$$\big\downarrow \wedge$$

$$\mathrm{hocolim}_{x,y\in I\times I} Map_{\mathcal{M}_n}[(S^{x\cup y})^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^x \wedge X))$$

$$\wedge \ S^y \wedge Y \wedge \Omega^\infty M_n(A)(S^0)]$$

$$\big\downarrow (\text{ right S}-\text{mod action})$$

$$\mathrm{hocolim}_{x,y\in I\times I} Map_{\mathcal{M}_n}[(S^{x\cup y})^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^{x\cup y} \wedge X \wedge Y) \wedge \Omega^\infty M_n(A)(S^0)]$$

$$\big\downarrow (\text{ right monoid action})$$

$$\mathrm{hocolim}_{x,y\in I\times I} Map_{\mathcal{M}_n}[(S^{x\cup y})^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^{x\cup y} \wedge X \wedge Y))]$$

$$\big\downarrow \cup$$

$$\mathrm{hocolim}_{z\in I} Map_{\mathcal{M}_n}[(S^z)^{\wedge *}, \widehat{cr}^* F(\Omega^\infty(\mathbf{n}_+ \wedge M)(S^z \wedge X \wedge Y))]$$

$$= \ L_n F(\Omega^\infty(\mathbf{n}_+ \wedge M))(X \wedge Y).$$

The derived tensor product of modules

Here we define what we mean by the derived tensor product of modules of FSPs. The construction is motivated by the definition of topological Hochschild homology for FSPs by M. Bökstedt in [Bö].

Let M be a left A–module and N a right A–module. We define $M\hat{\otimes}_A N$ as the S–module obtained from the simplicial S-module: $M\hat{\otimes}_A N(X)[n] =$

$$\mathrm{hocolim}_{x_i\in I^{n+1}} Hom(S^{x_0} \wedge \ldots \wedge S^{x_{n+1}}, \ M(S^{x_0}) \wedge A(S^{x_1}) \wedge \ldots \wedge A(S^{x_n}) \wedge N(S^{x_{n+1}}) \wedge X)$$

The simplicial operators are such that $M\hat{\otimes}_A N \cong TH(A; N \otimes_S M)$ and so one can look at [P-W] for example for an explicit description. However, in words, the *ith* face map takes the appropriate product which smashes the S^{x_i} coordinate with the $S^{x_{i+1}}$ coordinate. Then one uses the concatenation functor which sends $x_i \times x_{i+1}$ to $x_i \sqcup x_{i+1}$ and the induced map on homotopy colimits which this induces. Degeneracies are similarly defined, the *ith* degeneracy uses the identity in the $A(X^{i+1})$–coordinate.

Note (7.4): If M is a $B-A$–bimodule and N is a $A-C$–bimodule then $M\hat{\otimes}_A N$ is naturally a $B-C$–bimodule. We also note that multiplication produces a simplicial map $M\hat{\otimes}_A A \xrightarrow{\simeq} M$ of $B-A$–bimodules which is a simplicial homotopy equivalence (using the identity in A to produce a simplicial homotopy inverse). The functor $M\hat{\otimes}_A$ from $A-Mod$ to $B-Mod$ is a 1-excisive reduced homotopy functor which satisfies the limit axiom.

LEMMA (7.5): If F is an m–excisive reduced homotopy functor and G is an n–excisive reduced homotopy functor then $G \circ F$ is a (mn)–excisive reduced homotopy functor.

Proof: The following is not the shortest nor most sophisticated proof of this result– but it is elementary and fun. First observe that the result is trivially true if G is 1-excisive and hence by induction it suffices to do the case when G is homogeneous of degree n. Since homotopy orbits preserve "excisiveness", it suffices to do the case when G is the diagonal of an n–multi-linear functor $\hat{G}$. By induction along each variable of $\hat{G}$ it suffices to show that the composite functor $\hat{G}(\hat{F}_1, \ldots, \hat{F}_n)$ where each F_i is the diagonal of a k_i–multi-linear functor is degree Σk_i. This last fact, however, follows immediately from 3.4 of [Cal2] since the composite functor is itself the diagonal of a Σk_i–multi-linear functor.

Let F be a functor from $A-Mod$ to $B-Mod$. Then $L_nF(\Omega^\infty(\mathbf{n}_+ \wedge A)) = L_nF(\Omega^\infty A^n)$ is a $B - L_nS[M_n(A)]$–bimodule and $L_nS[Hom(\mathbf{n}_+,\ )]$ is a left $L_nS[M_n(A)]$–module. We define the natural transformation

$$\eta : L_nF(\Omega^\infty A^n)\hat{\otimes}_{L_nS[M_nA]} L_nS[Hom(\mathbf{n}_+,\ )] \longrightarrow L_nF(\Omega^\infty\ )$$

by composition.

THEOREM (7.6): For $F \in Hom_0^{st(n)}(A - Mod, B - Mod)$, the natural transformation η is an equivalence. Using η, one obtains a one-to-one correspondence between the homotopy category of $Hom_0(A - Mod, B - Mod)[\leq n]$ (n–excisive, reduced homotopy functors) and the homotopy category of $B - L_nS[M_n(A)]$-bimodules.

By theorem 6.2, to show η is an equivalence it suffices to show that η_{A^n} is an equivalence and that

$$G = L_nF(\Omega^\infty A^n)\hat{\otimes}_{L_nS[M_nA]} L_nS[Hom(\mathbf{n}_+,\ )]$$

is n–excisive and satisfies the limit axiom. We see that η_{A^n} is an equivalence by note 7.4 and that G is an n–excisive reduced homotopy functor by theorem 4.4 and lemma 7.5

(since $L_nS[Hom(\mathbf{n}_+, \ )]$ is the composite of a 1-excisive and an n–excisive functor). The limit axiom for G is immediate from the definition (using the fact that homotopy colimits commute).

(7.6) Compositions as tensor products

In this part all functors are reduced homotopy functors which satisfy the limit axiom. By lemma 7.4, the composition of a degree n functor F from A-modules to B-modules composed with a degree m functor G from B-modules to C-modules is a degree mn functor $G \circ F$ from A-modules to C-modules. We define a pairing of $B - L_nS[M_n(A)]$-bimodules with $C - L_mS[M_m(B)]$-bimodules which recovers composition (up to natural equivalence) with the classification result of theorem 7.5.

We first observe the following sequence of equivalences,

$$(G \circ F)(A^{mn})\hat{\otimes}_{L_{\mathbf{mn}}S[M_{mn}(A)]} L_{mn}S[Hom_A(\mathbf{mn}_+, \ )] \xrightarrow{\simeq} G \circ F(\ )$$

$$F(A^n)\hat{\otimes}_{L_nS[M_n(A)]} L_nS[Hom_A(\mathbf{n}_+, A^{mn})] \xrightarrow{\simeq} F(A^{mn})$$

$$\begin{array}{c} G(A^m)\hat{\otimes}_{L_mS[M_m(A)]} L_mS[Hom_B(\mathbf{m}_+, F(A^n)\hat{\otimes}_{L_nS[M_n(A)]} L_nS[Hom_A(\mathbf{n}_+, A^{mn})])] \\ \downarrow \simeq \\ G(A^m)\hat{\otimes}_{L_mS[M_m(A)]} L_mS[Hom_B(\mathbf{m}_+, F(A^{mn}))] \\ \downarrow \simeq \\ G(F(A^{mn})) = G \circ F(A^{mn}). \end{array}$$

Thus, if we define Φ to be the homotopy functor from $B - L_nS[M_n(A)]$-bimodules to $L_mS[M_m(B)] - L_{mn}S[M_{mn}(A)]$-bimodules by

$$\Phi(X) = L_mS[Hom_B(\mathbf{m}_+, X\hat{\otimes}_{L_nS[M_n(A)]} L_nS[Hom_A(\mathbf{n}_+, A^{nm})])]$$

then we can define a pairing

$$(B - Mod - L_nS[M_n(A)]) \times (C - Mod - L_mS[M_m(B)]) \ \xrightarrow{\mu} \ (C - Mod - L_{mn}S[M_{mn}(A)])$$

$$X \ \times \ Y \ \mapsto Y\hat{\otimes}_{L_mS[M_m(B)]}\Phi(X)$$

such that μ corresponds up to natural equivalence with composition in theorem 7.5.

We now examine the functor Φ in slightly more detail. Recall that $\Omega^\infty A(S^0)$ is a pointed (unital) monoid and for M an A-module, $\Omega^\infty M(S^0)$ is a left $\Omega^\infty A(S^0)$-space. In particular, it makes sense to talk about the $m \times n$–matrices of $\Omega^\infty A$ and

$$L_{mn}S[Hom_A(\mathbf{mn}_+, A^{mn})]$$

$$\Big\downarrow \simeq$$

$$S \wedge \text{hocolim}_{x\in I}\ Map_{\mathcal{M}_n}((S^x)^{\wedge *}, [S^x \wedge M_{mn,mn}(\Omega^\infty A(S^0))]^{\wedge *})$$

$$\Big\downarrow \cong$$

$$S \wedge \text{hocolim}_{x\in I}\ Map_{\mathcal{M}_n}((S^x)^{\wedge *}, (S^x \wedge M_{m,m}[M_{n,n}(\Omega^\infty A(S^0))])^{\wedge *}).$$

For X a $B - L_m S[M_n(A)]$-bimodule, $[\Omega^\infty X(S^0)]^{\times m}$ is a left $M_{m,m}[M_{n,n}(\Omega^\infty A(S^0))]$ space and

$$\Phi(X) \simeq S \wedge \text{hocolim}_{x\in I}\ Map_{\mathcal{M}_n}((S^x)^{\wedge *}, (S^x \wedge [\Omega^\infty X(S^0)]^{\times n})^{\wedge *})$$

as a $L_m S[M_m(B)] - L_{mn} S[M_{mn}(A)]$-bimodule.

Appendix A: Mapping spaces of diagrams

Recall that we let $\mathcal{M}_n$ be the dual of the category with one object $\mathbf{m} = \{1, \ldots, m\}$ for each $1 \leq m \leq n$ and morphisms the surjective set maps. The category of $\mathcal{M}_n$–diagrams of pointed simplicial sets is simply the category of functors from $\mathcal{M}_n$ to pointed simplicial sets with morphisms the natural transformations. It is a closed simplicial model category in the sense of Quillen ([Qui], Ch. II) and we refer the reader to [D-K] for the relevant definitions.

Following [D-K], 2.4 we call a map $f : X \to Y$ of $\mathcal{M}_n$–diagrams *free* if, for every object $\mathbf{m} \in \mathcal{M}_n$ the map $f(\mathbf{m})$ is an injection and if there exists a set B of simplices of Y such that

- i) no simplex of B is in the image of f
- ii) B is closed under degeneracy operators
- iii) for every object $\mathbf{m} \in \mathcal{M}_n$ and every simplex $y \in Y(\mathbf{m})$ which is not in the image of $f(\mathbf{m})$ there is a unique simplex $b \in B$ and a unique map $\alpha \in \mathcal{M}_n$ such that $Y(\alpha)b = y$.

The cofibrations of $\mathcal{M}_n$–diagrams are exactly the free maps and their retracts.

Let K be a pointed simplicial set. We write $K^{\wedge *}$ for the functor from $\mathcal{M}_n$ to pointed simplicial sets defined by

$$\mathbf{m} \mapsto K^{\wedge m}$$

$$f : \mathbf{m} \longrightarrow \mathbf{t} \;\Rightarrow\; f^*(x_1 \wedge \cdots \wedge x_t) = (x_{f(1)} \wedge \cdots \wedge x_{f(n)}).$$

LEMMA (A.1): The functor $K^{\wedge *}$ is a free object in the category of $\mathcal{M}_n$–diagrams and hence cofibrant.

Proof: For $\mathbf{m} \in \mathcal{M}_n$, let $C(\mathbf{m})$ be the subset of simplicies $(x_1 \ldots, x_m)$ of $K^{\wedge m}$ such that $x_i \neq x_j$ if $i \neq j$. Let $B(\mathbf{m})$ be any subset of $C(\mathbf{m})$ not containing the basepoint and with exactly one element for each Σ_m–orbit type and closed under degeneracy operators (for example, first choose one element of each Σ_m–orbit class of all the non-degenerate simplicies in $C(\mathbf{m})$ and then extend by degeneracies). If we let f be the map from the initial object $*^{\wedge *}$ to $K^{\wedge *}$, then using the set B we have constructed we see that f is free and hence $K^{\wedge *}$ is cofibrant.

Let $\Delta K^{\wedge n}$ be the *fat diagonal* of $K^{\wedge n}$. Thus,

$$\Delta K^{\wedge n} = \{(x_1, \ldots, x_n) | x_i = x_j \text{ for some } \; i \neq j\}$$

We let $\Delta_n K^{\wedge *}$ be the $\mathcal{M}_n$ diagram determined by

$$\Delta_n K^{\wedge m} = \begin{cases} K^{\wedge m} & \text{if } m \neq n \\ \Delta K^{\wedge n} & \text{if } m = n \end{cases}$$

Corollary (A.2): The map of $\mathcal{M}_n$–diagrams, i, from $\Delta_n K^{\wedge *}$ to $K^{\wedge *}$ given by inclusions is free and hence a cofibration.

Proof: The set $B = B(\mathbf{n})$ from lemma A.1 is easily seen to satisfy conditions i)–iii) with respect to the inclusion map i and hence the result.

Following [D-K] once again, we define the *twisted arrow category* $a\mathcal{M}_n$ of $\mathcal{M}_n$ to be the category which has as objects the maps of $\mathcal{M}_n$ and morphisms $(\mathbf{m}_0 \to \mathbf{m}_1) \to (\mathbf{m}'_0 \to \mathbf{m}'_1)$ the commutative diagrams (horizontal maps go in different directions).

$$\begin{array}{ccc} \mathbf{m}_0 & \longleftarrow & \mathbf{m}'_0 \\ \downarrow & & \downarrow \\ \mathbf{m}_1 & \longrightarrow & \mathbf{m}'_1 \end{array}$$

Given two $\mathcal{M}_n$ diagrams X and Y of pointed spaces or pointed simplicial sets, we can form the evident $a\mathcal{M}_n$–diagram $Hom_a(X, Y)$ and

$$Hom_{\mathcal{M}_n}(X, Y) \cong lim_{a\mathcal{M}_n}\ Hom_a(X, Y).$$

We also note that if X is a diagram of pointed simplicial sets and Y is a diagram of pointed spaces then we have a natural isomorphism (by adjointness, see for example [May], 16.1)

$$(1) \qquad Hom_{\mathcal{M}_n}(|X|, Y) \cong Hom_{\mathcal{M}_n}(X, Sing(Y))$$

Proposition (A.3, after ([D-K], 3.3): For K a pointed simplicial set and Y a $\mathcal{M}_n$ diagram of pointed spaces, the natural map from

$$Hom_{\mathcal{M}_n}(|K|^{\wedge *}, Y) \longrightarrow \mathrm{holim}_{a\mathcal{M}_n} Hom_a(|K|^{\wedge *}, Y)$$

is a weak equivalence. In particular, $Hom_{\mathcal{M}_n}(|K|^{\wedge *},\ )$ preserves weak equivalences (a map of diagrams is a weak equivalence if it is a weak equivalence when evaluated at each object).

Proof: By (1), it suffices to establish the result for $K^{\wedge *}$ and $Sing(Y)$. By lemma A.1 we know that $K^{\wedge *}$ is a cofibrant object. Since $Sing(Y(\mathbf{m}))$ is fibrant for each $\mathbf{m} \in \mathcal{M}_n$, $Sing(Y)$ is fibrant and the result follows from theorem 3.3 of [D-K].

Now we wish to examine the fiber of the restriction map from $Hom_{\mathcal{M}_n}(|K|^{\wedge *}, Y)$ to $Hom_{\mathcal{M}_{n-1}}(|K|^{\wedge *}, Y)$ obtained from the inclusion of the subcategory $\mathcal{M}_{n-1}$ into $\mathcal{M}_n$. We first note that this map factors naturally as

$$(2) \qquad Hom_{\mathcal{M}_n}(|K|^{\wedge *}, Y) \xrightarrow{i^*} Hom_{\mathcal{M}_n}(\Delta_n |K|^{\wedge *}, Y) \cong Hom_{\mathcal{M}_{n-1}}(|K|^{\wedge *}, Y)$$

where i is the inclusion map of diagrams in corollary A.2.

LEMMA (A.4): For K a pointed simplicial set and Y a $\mathcal{M}_n$–diagram of pointed spaces, there is a natural fibration sequence

$$Hom_{\Sigma_n}\left(\frac{|K|^{\wedge n}}{\Delta K^{\wedge n}}, Y(\mathbf{n})\right) \xrightarrow{q^*} Hom_{\mathcal{M}_n}(|K|^{\wedge *}, Y) \xrightarrow{res} Hom_{\mathcal{M}_{n-1}}(|K|^{\wedge *}, Y).$$

The action Σ_n is given by the full subcategory of $\mathcal{M}_n$ generated $\mathbf{n}$. The map q^* is induced by the quotient and res is the map given by restriction to a subcategory.

Proof: Once again, using the natural isomorphism of (1), we can do the result in the simplicial setting. By corollary A.2, the fact that $Sing(Y)$ is fibrant and the factorization in (2), the restriction map is a (Serre) fibration with fiber $Hom_{\mathcal{M}_n}(\frac{K^{\wedge n}}{\Delta K^{\wedge n}}, Sing(Y)(\mathbf{n}))$ which is naturally homeomorphic to $Hom_{\Sigma_n}\left(\frac{K^{\wedge n}}{\Delta K^{\wedge n}}, Y(\mathbf{n})\right)$ and hence the result.

Observation (A.5): An equivalent way of reformulation lemma A.4 is that the following natural commuting diagram is both a strict pull-back and Cartesian (a homotopy pull-back) as i^* is a fibration by lemma A.2:

$$\begin{array}{ccc} Hom_{\mathcal{M}_n}(|K|^{\wedge *}, Y) & \xrightarrow{res} & Hom_{\Sigma_n}(|K|^{\wedge n}, Y(\mathbf{n})) \\ \big\downarrow res & & \big\downarrow i^* \\ Hom_{\mathcal{M}_{n-1}}(|K|^{\wedge *}, Y) & \xrightarrow{\alpha} & Hom_{\Sigma_n}(\Delta|K|^{\wedge n}, Y(\mathbf{n})) \end{array}$$

where α is the natural composite

$$Hom_{\mathcal{M}_{n-1}}(|K|^{\wedge *}, Y) \cong Hom_{\mathcal{M}_n}(\Delta_n|K|^{\wedge *}, Y) \xrightarrow{res} Hom_{\Sigma_n}(\Delta|K|^{\wedge n}, Y(\mathbf{n})).$$

Appendix B: The Tate Map

Our goal in this section is to establish the Tate Map and a couple of results by T. Goodwillie in the appendix to his MSRI notes [MSRI]. Since these MSRI notes are not published, in this section we reproduce what is needed from them (B.2 and B.3 below) for our purposes in section 2. We have modified some of the constructions found in [MSRI] to make the proofs more transparent. We have chosen to write this appendix using the terminology of a functor with stabilization with G-action. The translation to $E \in \mathcal{S}pec_b$ with G–action is straightforward.

Let G be a group. Recall that for X a (pointed) space with G–action, we define the *homotopy orbit* space of X to be $X_{hG} = X \wedge_G EG_+ = (X \wedge EG_+)_G$. We recall that if $f : X \to Y$ is an n–connected G–equivariant map then f_{hG} is also n–connected. We note that if F is a functor with stabilization with G–action, then $X \mapsto F(X)_{hG}$ is again naturally a functor with stabilization.

We define the *homotopy fixed-point* space of X to be $X^{hG} = Map_G(EG_+, X) = Map_*(EG_+, X)^G$. If $f : X \to Y$ is a G–equivariant map which is also an equivalence, then f^{hG} is also an equivalence but $(\ )^{hG}$ does not preserve connectivity in general.

Definition B.1: Let G be a group and F a functor with stabilization with G-action. We define the *homotopy orbits* of F to be the functor with stabilization

$$F_{hG} = \Omega^\infty(X \mapsto [F(X)]_{hG})$$

and the *homotopy fixed–points* of F to be the functor with structure

$$F^{hG} = \Omega^\infty(X \mapsto [\Omega^\infty F(X)]^{hG}).$$

Important Remark: If F_* is a simplicial functor with stabilization with G–action, then the commuting diagram

$$\begin{array}{ccc} |(F_*)_{hG}| & \xrightarrow{\simeq} & \mathrm{hocolim}_{\Delta^{op}}[(F_*)_{hG}] \\ \downarrow \simeq & & \downarrow \cong \\ |(F_*)|_{hG} & \xrightarrow{\simeq} & (\mathrm{hocolim}_{\Delta^{op}}[F_*])_{hG}) \end{array}$$

shows that homotopy orbits commute with realizations. However, the natural map

$$|(F_*)^{hG}| \longrightarrow |F_*|^{hG}$$

is not an equivalence in general. That is, homotopy fixed points do not commute with realizations.

The Tate Map

For G a finite group, the *Tate map* is a chain of natural maps of functors with structure from F_{hG} to F^{hG} which we now wish to define. But first, we establish a sequence of natural equivalences

$$(G_+ \wedge F)_{hG} \simeq \Omega^\infty F \simeq (G_+ \wedge F)^{hG}$$

For X a G–space, we let γ be the G–equivariant map

$$G_+ \wedge X \cong \bigvee_G X \xrightarrow{inc} \prod_G X \cong Map\ (G_+, X)$$

that is:

$$\gamma(g \wedge x)(u) = \begin{cases} x & \text{if } g = u \\ * & \text{otherwise} \end{cases}$$

Thus, if X is k–connected, then γ is $(2k-1)$–connected by Blakers-Massey and we obtain the diagram:

$$(1)\qquad \begin{array}{ccccc} (G_+\wedge X)_G & \xleftarrow{\cong} & X & \xrightarrow{\cong} & Map(G_+,X)^G \\ \uparrow{\simeq} & & & & \downarrow{\simeq} \\ (G_+\wedge X)_{hG} & & & & Map(G_+,X)^{hG} \\ \downarrow{\gamma_{hG}} & & & & \uparrow{\gamma^{hG}} \\ Map(G_+,X)_{hG} & & & & (G_+\wedge X)^{hG} \end{array}$$

The map γ_{hG} is $(2k-1)$–connected but we do not know anything about γ^{hG}. However, if F is functor with stabilization with G–action then for all X we know that the map

$$(2)\qquad \begin{array}{c} [\text{hocolim } \Omega^n(G_+\wedge F(\Sigma^n X))]^{hG} \\ \downarrow{\gamma^{hG}} \\ [\text{hocolim } \Omega^n Map(G_+, F(\Sigma^n X)]^{hG} \end{array}$$

is an equivalence. We also note that since G is finite, the natural G–equivariant map

$$(3)\qquad \text{hocolim } Map(G_+,\Omega^n F(\Sigma^n X)) \xrightarrow{\simeq} Map(G_+, \text{hocolim } \Omega^n F(\Sigma^n X))$$

is an equivalence. Thus, we can assemble all these remarks to obtain the following sequence of natural equivalences of spaces (a symbol in "( )" indicates the previous statement which implies the map is an equivalence)

$$(A)\qquad \begin{array}{rc} (G_+\wedge F)_{hG}(X) = & \text{hocolim } \Omega^n[(G_+\wedge F(\Sigma^n X)_{hG}] \\ & \downarrow{\simeq\ (1)} \\ \Omega^\infty F(X) = & \text{hocolim } \Omega^n F(\Sigma^n X) \\ & \downarrow{\simeq\ (1)} \\ & [\text{Map } (G_+, \text{hocolim}\Omega^n F(\Sigma^n X)]^{hG} \\ & \uparrow{\simeq\ (3)} \\ & [\text{hocolim } \Omega^n Map(G_+, F(\Sigma^n X))]^{hG} \\ & \uparrow{\simeq\ (2)} \\ (G_+\wedge F)^{hG}(X) = & [\text{hocolim } \Omega^n(G_+\wedge F(\Sigma^n X))]^{hG} \end{array}$$

These assemble (as X varies) into a natural sequence of equivalences of functors with stabilization with G–action.

Using the G–equivariant equivalence $EG_+\wedge X \xrightarrow{\simeq} X$, we obtain natural equivalences on G–homotopy orbits and homotopy fix points. We can obtain EG_+ as the realization of a simplicial G–set $[q]\mapsto \bigwedge^{q+1} G_+$ (the simplicial path space of the bar construction for G). Thus,

$$EG_+\wedge F \ \cong_G \ |[q]\mapsto \bigwedge^{q+1} G_+\wedge F|$$

Definition B.2: (T. Goodwillie) The *Tate "map"* is the following natural diagram:

$$
\begin{array}{c}
F_{hG}\\
\uparrow \simeq\\
(EG_+ \wedge F)_{hG}\\
\downarrow \cong\\
|[q]\mapsto (\bigwedge^{q+1} G_+ \wedge F)|_{hG}\\
\uparrow \simeq\\
|[q]\mapsto (\bigwedge^{q+1} G_+\wedge F)_{hG}|\\
\\
\simeq\ (A)\\
\\
|[q]\mapsto (\bigwedge^{q+1} G_+\wedge F)^{hG}|\\
\downarrow\\
|[q]\mapsto (\bigwedge^{q+1} G_+\wedge F)|^{hG}\\
\downarrow\cong\\
|EG_+\wedge F|^{hG}\\
\downarrow\simeq\\
F^{hG}
\end{array}
$$

There is one case where the Tate map is easily seen to be an equivalence: when E is a functor with stabilization with G-action and $F = G_+ \wedge E$. This follows from the commuting diagram (using the projection maps π)

$$
(Free)\qquad
\begin{array}{ccc}
|[q]\mapsto [\bigwedge^{q+1} G_+ \wedge (G_+\wedge E)]_{hG}| & \overset{\pi(\simeq)}{\longrightarrow} & [G_+\wedge E]_{hG}\\
\\
\simeq\ (A) & & \\
\\
|[q]\mapsto [\bigwedge^{q+1} G_+ \wedge (G_+\wedge E)]^{hG}| & & \simeq\ (A)\\
\downarrow & & \\
|[q]\mapsto [\bigwedge^{q+1} G_+ \wedge (G_+\wedge E)]|^{hG} & \overset{\pi(\simeq)}{\longrightarrow} & [G_+\wedge E]^{hG}
\end{array}
$$

Proposition B.3: (T. Goodwillie) The Tate map for F is an equivalence if F is either $U \wedge E$ or Map (U, E), where E is a functor with stabilization with G–action and U is a pointed finite free G–space (i.e., a simplicial G–set with finitely many nondegenerate non-basepoint simplicies permuted freely by G).

The proof is by induction over skeleta; the cells attached at stage n are dealt with by applying $(Free)$ to the case $G_+ \wedge (Y \wedge \bigvee^t S^n)$.

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