Math 231 Worksheet #2

Instructions. Put the first and last name of everyone in your workgroup at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. (a) Using differential notation, write out the product rule for differentiation of two function \( u \) and \( v \)
   \[ d(uv) = \]
   
   Solutions:
   \[ d(uv) = du \cdot v + u \cdot dv \]

   (b) Write out the formula for integration by parts.
   \[ \int u dv = \]
   
   Solutions:
   \[ \int u dv = u \cdot v - \int v du \]

   (c) Explain how the product rule for differentiation leads to the formula for integration by parts.
   
   Solutions:
   Integrating the product rule:
   \[
   \int d(uv) = \int v du + \int u dv \\
   \int u dv = \left( \int d(uv) \right) - \int v du \\
   = u \cdot v - \int v du
   \]

2. Integrate by parts using the indicated substitution:
   (a) \[ \int \frac{x}{\cos^2(x)} \, dx \quad u = x \]

   Solutions:
   We choose \( u = x, \ du = dx, \ dv = \frac{dx}{\cos^2(x)}, v = \tan(x) \)

   \[
   \int \frac{x}{\cos^2(x)} \, dx = x \tan x - \int \tan x \, dx \\
   = x \tan x + \ln |\cos x| + C
   \]
(b) $\int x^3 e^{x^2} \, dx \quad u = x^2$

Solutions:
We choose $u = x^2$, $du = 2x \, dx$, $dv = x e^{x^2} \, dx$, $v = \frac{1}{2} e^{x^2}$

$$\int x^3 e^{x^2} \, dx = x^2 \frac{1}{2} e^{x^2} - \int \frac{1}{2} e^{x^2} 2x \, dx$$
$$= \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} \, dx$$
$$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C$$

3. Integrate by parts:
(a) $\int \arctan(x) \, dx$

Solutions:
We choose $u = \arctan(x)$, $du = \frac{1}{1+x^2} \, dx$, $dv = dx$, $v = x$

$$\int \arctan(x) \, dx = x \arctan(x) - \int \frac{1}{1+x^2} \, dx$$
$$= x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + C$$

(b) $\int (\ln x)^2 \, dx$

Solutions:
We choose $u = (\ln x)^2$, $du = \frac{2}{x} \ln x \, dx$, $dv = dx$, $v = x$

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x \, dx$$

We can do parts again with $u = \ln x$ and $dv = dx$ to get

$$(\ln x)^2 \, dx = x(\ln x)^2 - 2 \left( x \ln x - \int dx \right)$$
$$= x (\ln x)^2 - 2x \ln x + 2x + C$$

(c) $\int_1^2 s^2 \ln s \, ds$

Solutions:
We choose \( u = \ln s, \ du = \frac{1}{s} \ ds, \ dv = s^2 \ ds, \ v = \frac{1}{3} s^3 \)

\[
\int_1^2 s^2 \ln s \ ds = \left[ \frac{1}{3} s^3 \ln s \right]_1^2 - \frac{1}{3} \int_1^2 s^2 \ ds
\]
\[
= \left[ \frac{1}{3} s^3 \ln s \right]_1^2 - \frac{1}{3} \int_1^2 s^2 \ ds
\]
\[
= \left[ \frac{1}{3} s^3 \ln s - \frac{1}{9} s^3 \right]_1^2
\]
\[
= \left[ \frac{1}{9} s^3 (3 \ln s - 1) \right]_1^2
\]
\[
= \frac{1}{9} s^3 (3 \ln 2 - 1) - \frac{1}{9} s^3 (3 \ln 1 - 1)
\]
\[
= \frac{1}{9} (24 \ln 2 - 7)
\]

4. Evaluate the integral:
   (a) \( \int \sin^2 x \cos^3 x \ dx \)

   **Solutions:**

\[
\int \sin^2 x \cos^3 x \ dx = \int \sin^2 x(1 - \sin^2 x) \cos x \ dx
\]
\[
= \int \sin^2 x \cos x \ dx - \int \sin^4 x \cos x \ dx
\]

Substitute: \( u = \sin x, \ du = \cos x \ dx \)

\[
= \int u^2 \ du - \int u^4 \ du
\]
\[
= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C
\]
\[
= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C
\]

(b) \( \int \tan^3 \theta \sec \theta \ d\theta \)

**Solutions:**
We choose \( u = \sec \theta, \ du = \sec \theta \tan \theta \ d\theta \)
Also recall: \( \tan^2 \theta = \sec^2 \theta - 1 \)

\[
\int \tan^3 \theta \sec \theta \, d\theta = \int \tan^2 \theta \tan \theta \sec \theta \, d\theta \\
= \int (\sec^2 \theta - 1) \tan \theta \sec \theta \, d\theta \\
= \int (u^2 - 1) \, du \\
= \frac{u^3}{3} - u + C \\
= \frac{1}{3} \sec^3 \theta - \sec \theta + C
\]

(c) \( \int_0^{\pi} \sqrt{1 + \cos 2x} \, dx \)

**Solutions:**

Recall:

\[
\cos(2x) = 1 - 2\sin^2 x \\
1 + \cos(2x) = 2 - 2\sin^2 x = 2(1 - \sin^2 x) = 2\cos^2 x
\]

Then:

\[
\int_0^{\pi} \sqrt{1 + \cos(2x)} \, dx = \int_0^{\pi} \sqrt{2} \cos x \, dx = \left[ \sqrt{2} \sin x \right]_0^{\pi} = \sqrt{2} \left( \sin \frac{\pi}{2} - 0 \right) = \frac{\sqrt{2}}{2}
\]