1. (Dif. Eq). Power series can be used to solve differential equations. Suppose that a country has at time $t$ a population $p(t)$, and that the rate of change of the population is given by $p'(t) = 2p(t)$.

(a) If we represent the population by a power series $p_0 + p_1 t + p_2 t^2 + \ldots$, find from the differential equation a relation between the coefficients.

Solution:

We have:

$$p(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + \ldots$$

Then the differential equation can be rewritten as:

$$p'(t) = p_1 + 2p_2 t + 3p_3 t^2 + 4p_4 t^3 + \ldots$$

Comparing the coefficients:

$$1p_1 = 2p_0$$

$$2p_2 = 2p_1$$

$$3p_3 = 2p_2$$

Then the power series can be expressed as:

$$\sum_{n=0}^{\infty} 2^n \cdot t^n$$

(b) Find the power series if $p_0 = 1$.

Solution:

$$p_1 = \frac{1}{2} p_0 = 2$$

$$p_2 = \frac{1}{2} p_1 = 2$$

$$p_3 = \frac{1}{3} p_2 = \frac{4}{3}$$

$$p_4 = \frac{1}{4} p_3 = \frac{8}{15}$$

Then the power series can be expressed as:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot t^n$$

(c) Identify the power series as the Taylor expansion of some function $f(t)$.

Solution:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot t^n = \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} = e^{2t}$$

2. Approximate $f$ by a Taylor polynomial with degree $n$ at the number $a$. Then use Taylor’s Inequality to estimate the accuracy of the approximation when $x$ lies in the given interval.

$$f(x) = \sqrt{x}, \quad a = 4, \quad n = 2, \quad 4 \leq x \leq 4.2$$

Solution:
\[
f'(x) = \frac{1}{2} \cdot x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}}
\]
\[
f''(x) = -\frac{1}{4} \cdot x^{-\frac{5}{2}} = -\frac{1}{4x\sqrt{x}}
\]
\[
f^{(3)}(x) = \frac{1}{4} \cdot \frac{3}{2} \cdot x^{-\frac{7}{2}} = \frac{3}{8} \cdot \frac{1}{x^2\sqrt{x}}
\]
\[
T_2(x) = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2
\]
Then: \[
T_2(x) = 2 + \frac{1}{4} (x - 4) - \frac{1}{4 \cdot 4 \cdot 2} (x - 4)^2
\]
\[
\sqrt{x} \approx 2 + \frac{1}{4} (x - 4) - \frac{1}{64} (x - 4)^2
\]

Taylor Inequality:
\[
|R_2(x)| \leq \frac{M}{3!} |x - 4|^3 \quad \text{where } |f'''(x)| \leq M
\]
Because \( x \geq 4 \), \( x^{-\frac{5}{2}} \leq 4^{-\frac{5}{2}} \) and so: \[
f'''(x) = \frac{1}{4} \cdot \frac{3}{2} \cdot \frac{1}{x^2\sqrt{x}} \leq \frac{1}{4} \cdot \frac{3}{2} \cdot \frac{1}{4^2\sqrt{4}} = \frac{3}{4^4}
\]
Then \( M = \frac{3}{4^4} \)
Also \( 4 \leq x \leq 4.2 \), so \( 0 \leq x - 4 \leq 0.2 \)
Then:
\[
|R_2(x)| = \frac{3}{4^4} \cdot (0.2)^3 = 0.000015625
\]

3. An electric charge \( q \) at \( x = 0 \) produces an electric field \( E(x) = \frac{q}{x^2} \). If we have two opposite electric charges situated at \( x = -d \) (charge -q) and at \( x = 0 \) (charge +q), the total electric field produced at \( D \) is \( E(D) = \frac{q}{D^2} - \frac{q}{(D+d)^2} \).

(a) Find the expansion of \( \frac{1}{(D+d)^2} = \frac{1}{D^2(1+\frac{d}{D})^2} \) in positive powers of \( d/D \).

Solution:
\[
\frac{1}{(D+d)^2} = \frac{1}{D^2} \cdot \left(1 + \frac{d}{D}\right)^{-2} = \frac{1}{D^2} \cdot \sum_{n=0}^{\infty} \left(-2\right)^n \binom{n}{D}^n
\]
\[
\frac{1}{(D+d)^2} = \frac{1}{D^2} \cdot \left(1 - 2\left(\frac{d}{D}\right) + 3\left(\frac{d}{D}\right)^2 - 4\left(\frac{d}{D}\right)^3 + 5\left(\frac{d}{D}\right)^4 + \ldots\right)
\]
(b) Use this to find an approximation of \( E(D) \approx \frac{2qd}{D^3} \) for large values of \( D \). This approximation is used often in Physics.

Solution:
If we assume \( D \) to be large, then we can assume that \( \left(\frac{d}{D}\right)^2 \) is very small. So we truncate the above series expansion after the linear term:
\[
\frac{1}{(D+d)^2} \approx \frac{1}{D^2} \cdot \left(1 - 2\left(\frac{d}{D}\right)\right)
\]
Then:
\[ E(D) \approx \frac{q}{D^2} - q \cdot \frac{1}{(D + d)^2} = \frac{q}{D^2} - q \cdot \frac{1}{D^2} \cdot \left(1 - 2 \left(\frac{d}{D}\right)\right) \]
\[ E(D) \approx \frac{q}{D^2} - \frac{q}{D^2} + \frac{2qd}{D^3} \]
\[ E(D) \approx \frac{2qd}{D^3} \]