1. (Ratio test and Radius of convergence.)

(a) Consider a power series \( \sum_{n=0}^{\infty} a_n x^n \). Suppose that \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \). What is the radius of convergence of the power series?

**Solution:**

**Ratio Test:**

\[
\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = L \cdot |x|
\]

For the series to converge:

\[
L \cdot |x| < 1
\]

\[
|x| < \frac{1}{L}
\]

\[
-\frac{1}{L} < x < \frac{1}{L}
\]

Radius of convergence: \( \frac{1}{L} \)

(b) If for \( \sum_{n=0}^{\infty} a_n (x - 1)^n \) we have \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \), what is the radius of convergence?

**Solution:**

**Ratio Test:**

\[
\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{a_{n+1} (x - 1)^{n+1}}{a_n (x - 1)^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| \cdot |x - 1| = L \cdot |x - 1|
\]

For the series to converge:

\[
L \cdot |x - 1| < 1
\]

\[
|x - 1| < \frac{1}{L}
\]

Radius of convergence: \( \frac{1}{L} \)

(c) Consider \( a_n = (-1)^n \frac{n^2}{2^n} \). What is \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \)? What is the *interval* of convergence for \( \sum_{n=0}^{\infty} (-1)^n \frac{n^2}{2^n} (x - 2)^n \)? (So find the radius of convergence, and investigate what happens at the endpoints)

**Solution:**

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} \frac{(n+1)^2}{2^{n+1}}}{(-1)^n \frac{n^2}{2^n}} \right| = \left| \frac{n^2}{2^n} \cdot 2 \cdot \frac{2^{n+1}}{2^{n+1}} \right| = \frac{1}{2} \text{ as } n \to \infty
\]

\[
\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{1}{2} \cdot (x - 2) \right| = \frac{1}{2} \cdot |x - 2|
\]
For the series to converge:

\[ \frac{1}{2} \cdot |x - 2| < 1 \]
\[ |x - 2| < 2 \]
\[ -2 < x - 2 < 2 \]
\[ 0 < x < 4 \]

Radius of convergence: 2  
Endpoints: \( x = 0, x = 4 \)

Checking \( x = 0 \):

\[ \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n^2}{2^n} \cdot (-2)^n = \sum_{n=0}^{\infty} (-1)^{2n} \cdot \frac{n^2}{2^n} \cdot 2^n = \sum_{n=0}^{\infty} n^2 \text{ diverges} \]

Checking \( x = 4 \):

\[ \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n^2}{2^n} \cdot 2^n = \sum_{n=0}^{\infty} (-1)^n \cdot n^2 \text{ diverges} \]

So the interval of convergence is \( 0 < x < 4 \)

2. Suppose that \( c(x) = \sum_{n=0}^{\infty} c_n x^n \) converges for \( x = -4 \), but diverges for \( x = 6 \).

(a) What can you say about the convergence of \( c(x) \) at \( x = 2? \ x = -7? \)

**Solution:**

By Theorem 3 (page 743) there are only 3 possible ways of convergence. Thus if the series converges for \( x = -4 \), it will converge at least in the interval \([-4, 4]\) so it will converge for \( x = 2 \). Similarly, since the series diverges at \( x = 6 \), it will diverge for all \( |x| > 6 \) and hence diverge for \( x = -7 \).

(b) What can you say about the convergence of \( \sum_{n=0}^{\infty} c_n \)?

**Solution:**

This is the series when evaluated at 1 and since 1 \( \in [-4, 4] \) we know it absolutely converges.

(c) What can you say about the convergence of \( \sum_{n=0}^{\infty} (-1)^n c_n 3^n \)?

**Solution:** This is the series evaluated at \( -3 \in [-4, 4] \) and so absolutely converges.

3. Find the Radius and Interval of Convergence of the following power series:

(a) \( \sum_{n=1}^{\infty} \frac{x^n}{n} \)

**Solution:**

Ratio Test: \( \lim_{n \to \infty} \left| \frac{x^{n+1}}{n+1} \right| = \lim_{n \to \infty} \left| x \cdot \frac{n}{n+1} \right| = |x| < 1 \) for convergence

\( |x| < 1 \iff -1 < x < 1 \)

Radius of convergence: 1  
Endpoints: \( x = -1, x = 1 \)
Checking \( x = -1 \): \[ \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} \] Converges by Alternating Series Test

Checking \( x = 1 \): \[ \sum_{n=1}^{\infty} \frac{1}{n} \] Diverges

Interval of convergence: \([-1, 1)\)

(b) \( \sum_{n=0}^{\infty} x^n/n! \)

Solution:

Ratio Test: \( \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{x^n}{n!} \right| = \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0 \) Converges for all \( x \)

(c) \( \sum_{n=0}^{\infty} \sqrt{n} x^n \)

Solution:

Ratio Test: \( \lim_{n \to \infty} \left| \frac{\sqrt{n+1} \cdot x^{n+1}}{\sqrt{n} \cdot x^n} \right| = \lim_{n \to \infty} \sqrt{\frac{n+1}{n}} |x| = |x| < 1 \) for convergence

\(|x| < 1 \iff -1 < x < 1\)

Radius of convergence: 1

Endpoints: \( x = -1, x = 1 \)

Checking \( x = -1 \): \[ \sum_{n=0}^{\infty} (-1)^n \cdot \sqrt{n} \] Diverges

Checking \( x = 1 \): \[ \sum_{n=0}^{\infty} \sqrt{n} \] Diverges

Interval of convergence: \((−1, 1)\)

(d) \( \sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{2n+1} \)

Solution:

Ratio Test: \( \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{2(n+1)+1} \cdot \frac{(x-3)^n}{2n+1} \right| = \lim_{n \to \infty} \left| (x-3) \cdot \frac{2n+1}{2n+3} \right| = |x-3| < 1 \) to converge

\(|x-3| < 1 \iff -1 < x - 3 < 1 \iff 2 < x < 4\)

Radius of convergence: 1

Endpoints: \( x = 2, x = 4 \)

Checking \( x = 2 \): \[ \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x-3)^n}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} \] Converges by Comparison Test

Checking \( x = 4 \): \[ \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{2n+1} \] Converges by Alt. Series Test

Interval of convergence: \((2, 4]\)