Worksheet #6
Math 221

Instructions: Put the first and last name of everyone in your group at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

Some questions on this worksheet reference a computer-generated demonstration (access it through Moodle) to aid in conceptual understanding of the questions.

1 The limit definition of the derivative

1. Let \( f(x) = x^3 - 9x^2 + 11x - 21 \).

   (a) According to the demonstration, what is the slope of the tangent line at \((3, -42)\)?
   Ans. \(-16\).

   (b) Use the demonstration to determine the slope of the secant line passing through the points \((3, -42)\) and \((3 + h, f(3 + h))\) to complete the tables below.

   \[
   \begin{array}{c|c}
   h & \text{Slope of the secant line} \\
   \hline
   -1 & -15 \\
   -0.25 & -15.9375 \\
   -0.05 & -15.9975 \\
   \end{array}
   \]

   \[
   \begin{array}{c|c}
   h & \text{Slope of the secant line} \\
   \hline
   1 & -15 \\
   0.25 & -15.9375 \\
   0.05 & -15.9975 \\
   \end{array}
   \]

   (c) As \( h \to 0 \), how does the slope of the secant line compare to the slope of the tangent line to \( f(x) \) at the point \((3, -42)\)? Explain.
   Ans. As \( h \to 0 \), the slope of the secant line approaches the slope of the tangent line at the point \( a = 3 \).

   (d) What formula measures the slope of the secant line between the points \((a, f(a))\) and \((a + h, f(a + h))\)?
   Ans. \( \frac{f(a + h) - f(a)}{(a + h) - a} \).

   (e) Graphically, the derivative of a function at a point is equal to what?
   Ans. The slope of the tangent line at the point.

   (f) Explain how the above ideas lead to the limit definition of the derivative.
   Ans. Indeed, the slope of the tangent line is the derivative of the function at that point and in the limiting case the slope of the secant line is the derivative of the function at that point. Therefore, by part (d) and (e), we obtain that \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \).
2. Use the limit definition of the derivative to show that if \( g(x) = 4x^2 - 5x + 2 \), then \( g'(x) = 8x - 5 \).
   
   **Ans.**

\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{4(x + h)^2 - 5(x + h) + 2 - 4x^2 + 5x - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{4x^2 + 4h^2 + 8xh - 5x - 5h + 2 - 4x^2 + 5x - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{4h^2 + 8xh - 5h}{h} = \lim_{h \to 0} (4h + 8x - 5) = 8x - 5.
\]

2 Rules of differentiation

3. For this problem, let

\[
f(x) = c \cdot \left( x (x^2 - 1) \sin \left( \frac{3\pi x}{2} \right) (\sin(2\pi x) + \cos(\pi x)) + 0.5 \right), \quad 0.5 \leq c \leq 5.
\]

(a) Use the demonstration to determine \( f'(a) \) for the values of \( a \) and \( c \) below.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c = 0.5 )</th>
<th>( c = 1 )</th>
<th>( c = 2 )</th>
<th>( c = 3 )</th>
<th>( c = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>-4</td>
<td>-6</td>
<td>-10</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

(b) How does multiplying any function \( h(x) \) by \( c > 0 \) change the graph of the function? **Ans.** It stretches or shrinks the graph vertically.

(c) Based on the above, how does \( \frac{d}{dx} (c \cdot h(x)) \) relate to \( h'(x) \)? Explain why this makes sense.

**Ans.** \( \frac{d}{dx} (c \cdot h(x)) \) is \( h'(x) \) multiplied by the constant \( c \), which makes sense since the slope of the tangent line to the new stretched or shrunk function at a point is that constant times the slope of the original function.

4. Here, suppose that \( h(x) = 3f(x) - g(x) \). Using the chart below, determine \( h'(-2) \), \( h'(5) \), and \( h'(17) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>-3</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Solution:** \( h'(x) = 3f'(x) - g'(x) \). Therefore,

\[
h'(-2) = 3(-3) - 2 = -11; \quad h'(5) = 3 \cdot 5 - 4 = 11; \quad h'(17) = 3(-1) - 0 = -3.
\]
5. In this question, suppose that \( h(x) = f(x) \cdot g(x) \) for \( f(x) \) and \( g(x) \) as shown in the accompanying demonstration. Determine \( h'(0) \), \( h'(-1) \), and \( h'(1) \).

**Solution:** \( h'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x) \). Therefore,

\[
h'(0) = -5 \cdot 0 - 5 \cdot 10 = -50; \quad h'(-1) = 5 \cdot 9 - 4 \cdot 5 = 25; \quad h'(1) = 3(-3) + 0 = -9.
\]

6. Suppose that when ordering photo prints online, you resize one photo from 4 \times 6 to 8 \times 12 (inches). If it takes you exactly two seconds to drag one corner and resize the photo, at what rate was the area of the photo changing after 1 second? (There is an available demonstration to help visualize this scenario.)

**Solution:** Here, we’ll let \( x \) measure the horizontal side length and \( y \) measure the vertical side length. Both of these dimensions are changing over time, which are expressed by \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \). Because it takes 2 seconds to resize the photo, we have \( \frac{dx}{dt} = \frac{2\text{ in}}{\text{sec}} \) and \( \frac{dy}{dt} = \frac{3\text{ in}}{\text{sec}} \). What we want to know, however, is the rate at which the area was changing after 1 second. Area is \( A(t) = x(t) \cdot y(t) \), which means \( \frac{dA}{dt} = x'(t)y(t) + x(t)y'(t) \). After 1 second, \( x(1) = 6 \) and \( y(1) = 9 \). This means that

\[
\left. \frac{dA}{dt} \right|_{t=1} = 2 \cdot 9 + 6 \cdot 3 = 36 \text{ in}^2/\text{sec}.
\]

7. Suppose an ice cube that is 3 \times 3 \times 3 cm\(^3\) begins to melt in a way that it maintains the shape of a cube and that the resulting water forms a circular puddle (1 mm deep) about the cube. How fast is the area of the puddle increasing when the cube has shrunk to 2 \times 2 \times 2 cm\(^3\) if the radius of the puddle is increasing at a rate of 5 mm per minute at that instant? (Note: 1 cm = 10 mm.)

**Solution:** (There are two main assumptions in this problem to simplify the computations. First, we assume that the volume of ice is equivalent to the volume of water it produces. Second, we assume that the cube is sitting on top of the puddle of water.) From the problem statement, we know that \( \frac{dr}{dt} = \frac{5\text{ mm}}{\text{min}} \). The area of a circle is \( A = \pi r^2 \), so if we want to know the rate at which the area of the puddle is increasing, we want to know \( \frac{dA}{dt} \) which is equal to \( 2\pi r \frac{dr}{dt} \) (by the product rule). We know \( \frac{dr}{dt} \) for the instant when the cube has shrunk to 2 \times 2 \times 2, but we need to know how big the radius is at that point. To determine this, we need to calculate how much water has melted from the ice cube. Originally, the cube had 27 cm\(^3\) but has shrunk to 8 cm\(^3\), which means we have 19 cm\(^3\) = 1,900 mm\(^3\). Dividing this by the puddle’s depth, we have that the surface area of the puddle is 1,900 mm when the cube is 2 \times 2 \times 2. This means that \( 1,900 = \pi r^2 \) or \( r = \sqrt{\frac{1900}{\pi}} \). This allows us to answer the question and say that when the cube has shrunk to 2 \times 2 \times 2, we have

\[
\frac{dA}{dt} = 2\pi \left( \sqrt{\frac{1900}{\pi}} \text{ mm} \right) \cdot \left( 5 \frac{\text{mm}}{\text{min}} \right).
\]