Worksheet #24
Math 221

Instructions: Put the first and last name of everyone in your group at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

Some questions on this worksheet reference a computer-generated demonstration (access it through Moodle) to aid in conceptual understanding of the questions.

1. Write the integral you would use to calculate the area bounded by the curves \( y = x + 5 \) and \( y = x^2 - 1 \).

   Solution: In order to find the point of intersections for the two given curves,

   \[
   x + 5 = x^2 - 1 \implies x^2 - x - 6 = 0 \implies (x - 3)(x + 2) = 0 \implies x = 3, -2.
   \]

   When \( x = 0 \), \( 0 + 5 = 5 > -1 = 0^2 - 1 \)Therefore, the desired integral is

   \[
   \int_{-2}^{3} (x + 5) - (x^2 - 1) \, dx.
   \]

2. The area of the region bounded by \( y = \sqrt{x} \) and \( y = x^2 \) (shown in the image below) can be expressed as an integral with respect to \( y \) and an integral with respect to \( x \). What are these integrals?

   ![Diagram of the region bounded by \( y = \sqrt{x} \) and \( y = x^2 \).]

   Solution: For the limits for \( x \), we need to set the two functions equal to each other. i.e.

   \[
   \sqrt{x} = x^2 \implies x = x^4 \implies x(x^3 - 1) = 0 \implies x = 0, 1.
   \]
Therefore, the required integral is
\[ \int_0^1 (\sqrt{x} - x^2) \, dx. \]

Now w.r.t. \( y \):
\[ x = y^2; x = +\sqrt{y} \]
In a similar way as for \( x \), we obtain the limits for \( y \) from 0 to 1 and the desired integral is
\[ \int_0^1 (\sqrt{y} - y^2) \, dy. \]

3. Let \( \mathcal{R} \) be the region bounded by the curves \( y = x^2 - 2x \) and \( y = x \).

(a) Sketch the region \( \mathcal{R} \) between the two curves and identify any important points.

**Solution:** For the points of intersection:
\[ x^2 - 2x = x \Rightarrow x^2 - 3x = 0 \Rightarrow x(x - 3) = 0 \Rightarrow x = 0, 3. \]

(b) In each part below, set up (but do NOT evaluate) the definite integral that you would use to calculate the specified quantity.

i. The area of the region between the two curves.

**Solution:** For \( x \in [0, 3] \), we have \( x^2 - 2x < 2x \). Therefore, the area is given by
\[ \int_0^3 (x - (x^2 - 2x)) \, dx \]

ii. The volume of the object with base equal to \( \mathcal{R} \) and with circular cross sections (taken perpendicular to the \( x \)-axis).

**Solution:** Area of the circle is \( \pi r^2 \), here \( r = \frac{x - (x^2 - 2x)}{2} \). Therefore, the volume is
\[ \pi \int_0^3 \left( \frac{x - (x^2 - 2x)}{2} \right)^2 \, dx. \]

iii. The volume of the object with base equal to \( \mathcal{R} \) and with square cross sections.

**Solution:**
\[ \int_0^3 (x - (x^2 - 2x))^2 \, dx. \]
iv. The volume of the object with base equal to $\mathcal{R}$ and with equilateral triangles for cross sections.

**Solution:** For an equilateral triangle with side length $s$, the height is $\frac{\sqrt{3}s}{2}$ and so the area is $\frac{1}{2} \cdot \frac{\sqrt{3}s}{2} = \frac{\sqrt{3}}{4} s^2$.

$$\frac{\sqrt{3}}{4} \int_{0}^{3} (x - (x^2 - 2x))^2 \, dx.$$  

4. Let $\mathcal{R}$ be the region bounded by the curves $y = \frac{24}{x^2 + 2}$ and $y = x^2$.

(a) Sketch the region $\mathcal{R}$ between the two curves and identify any important points.

**Solution:** For the points of intersection:

$$\frac{24}{x^2 + 2} = x^2 \Rightarrow x^4 + 2x^2 - 24 = 0 \Rightarrow (x^2 + 6)(x^2 - 4) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$$  

(b) In each part below, set up (but do NOT evaluate) the definite integral that you would use to calculate the specified quantity.

i. The area of the $\mathcal{R}$.

**Solution:**

$$\int_{-2}^{2} \left( \frac{24}{x^2 + 2} - x^2 \right) \, dx.$$  

ii. The volume of the object with base equal to $\mathcal{R}$ and with semi-circular cross sections (taken perpendicular to the $x$-axis).

**Solution:**

$$\frac{\pi}{2} \int_{-2}^{2} \left( \frac{24/(x^2 + 2) - x^2}{2} \right)^2 \, dx.$$  

iii. The volume of the object with base equal to $\mathcal{R}$ and with square cross sections.

**Solution:**

$$\int_{-2}^{2} \left( \frac{24}{x^2 + 2} - x^2 \right)^2 \, dx.$$  

iv. The volume of the object with base equal to $\mathcal{R}$ and cross sections that look like triangles with height equal to half the base.

**Solution:**

$$\frac{1}{4} \int_{-2}^{2} \left( \frac{24}{x^2 + 2} - x^2 \right)^2 \, dx.$$  

3
5. In this problem, you will establish that the volume of a sphere with radius \( R \) is equal to \( \frac{4}{3} \pi R^3 \). To do so, consider the region in the \( xy \)-plane that is bounded by the curves \( y = \sqrt{R^2 - x^2} \) and \( y = -\sqrt{R^2 - x^2} \).

(a) Sketch this region and set up the definite integral to calculate the area of \( R \). Use geometry to evaluate your integral. (In Calc II, you will learn how to directly evaluate this integral and show that the area of a circle is \( \pi R^2 \).)

Solution:

\[
\int_{-R}^{R} \left( \sqrt{R^2 - x^2} - (-\sqrt{R^2 - x^2}) \right) \, dx = 2 \int_{-R}^{R} \sqrt{R^2 - x^2} \, dx = 2R \int_{-R}^{R} \sqrt{1 - x^2/R^2} \, dx 
\]

(1)

Let \( x/R = u \), then \( dx = R \, du \) and for \( x = -R, u = -1 \), for \( x = R, u = 1 \). Therefore,

\[
2R \int_{-R}^{R} \sqrt{1 - x^2/R^2} \, dx = 2R^2 \int_{-1}^{1} \sqrt{1 - u^2} \, du = 4R^2 \int_{0}^{1} \sqrt{1 - u^2} \, du
\]

Now, from Problem 4(a) in Worksheet 22, we conclude that

\[
4R^2 \int_{0}^{1} \sqrt{1 - u^2} \, du = 4R^2 \left( \frac{\pi}{4} - \cos^{-1} 1 \right) = \pi R^2 - 4R^2 \times 0 = \pi R^2
\]

OR In order to compute the integral \( 2 \int_{-R}^{R} \sqrt{R^2 - x^2} \, dx \) in (1), make the substitution \( x = R \sin \theta \). Then \( dx = R \cos \theta \, d\theta \), and for \( x = -R, \sin \theta = -1 \Rightarrow \theta = -\pi/2 \) and similarly for \( x = R, \theta = \pi/2 \). Therefore,

\[
2 \int_{-R}^{R} \sqrt{R^2 - x^2} \, dx = 2R^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 \theta} \cos \theta \, d\theta
\]

\[
= R^2 \int_{-\pi/2}^{\pi/2} 2 \cos^2 \theta \, d\theta = R^2 \int_{-\pi/2}^{\pi/2} (\cos(2\theta) + 1) \, d\theta
\]

\[
= R^2 (\sin(2\theta)/2 + \theta) |_{-\pi/2}^{\pi/2} = \pi R^2.
\]

(b) Set up AND calculate the volume of the object with base \( R \) and circular cross sections (taken perpendicular to the \( x \)-axis).

Solution:

\[
\int_{-R}^{R} \pi(2\sqrt{R^2 - x^2}/2)^2 \, dx = \pi \int_{-R}^{R} (R^2 - x^2) \, dx = \pi (R^2 x - x^3/3) |_{-R}^{R}
\]

\[
= \pi (R^3 - R^3/3 - [-R^3 + R^3/3]) = 2\pi (R^3 - r^3/3)
\]

\[
= \frac{4}{3} \pi R^3.
\]