Worksheet #22
Math 221

1. Compute \( \frac{d}{ds} \int_{\cos s}^{s^2 + 2} \frac{1}{x^2 + 1} \, dx. \)

**Solution:** Let \( f(t) = \frac{1}{t^2 + 1}. \) Then, by the fundamental theorem of calculus, we have

\[
\frac{d}{ds} \int_{\cos s}^{s^2 + 2} \frac{1}{x^2 + 1} \, dx = f(s^2 + 2) \times \frac{d}{ds} (s^2 + 2) - f(\cos s) \times \frac{d}{ds} (\cos s)
\]

\[
= \frac{1}{(s^2 + 2)^2 + 1} \times (2s) - \frac{1}{\cos^2 s + 1} \times (-\sin s)
\]

2. A point is moving along the \( x \)-axis with acceleration \( 45t + 1 \) at time \( t \). If its initial velocity is 27, how far does the particle move from time \( t = 1 \) to time \( t = 3 \)?

**Solution:** Let the acceleration be \( a(t) \). Then the velocity \( v(t) \) is given by \( \int a(t) \, dt. \) Therefore,

\[
v(t) = \int (45t + 1) \, dt = \frac{45t^2}{2} + t + c.
\]

Since \( v(0) = 27 \), we conclude \( c = 27 \). For the distance covered between time \( t = 1 \) and \( t = 3 \), we need:

\[
\int_{1}^{3} \left( \frac{45t^2}{2} + t + 27 \right) \, dt = \left( \frac{45t^3}{6} + \frac{t^2}{2} + 27t \right) \bigg|_{1}^{3}
\]

\[
= \frac{45}{6} \times 27 + \frac{9}{2} + 27 \times 3 - \left( \frac{45}{6} + \frac{1}{2} + 27 \right)
\]

3. A patient is experiencing blood loss at a rate of \( \frac{2}{1 + t^2} \) pints per minute at time \( t \) (as \( t \) increases, the blood loss slows due to decreased pressure). Approximately how long before the patient loses 3 pints of blood after \( t = 0 \)?

**Solution:** Let \( f(t) = \frac{2}{1 + t^2} \). Then, we want to find \( a \) so that \( \int_{0}^{a} f(t) \, dt = 3 \). i.e.

\[
3 = \int_{0}^{a} \frac{2}{1 + t^2} \, dt = 2 \arctan t \bigg|_{0}^{a} = 2 \arctan a - 2 \arctan 0 = 2 \arctan a
\]

\[
\Rightarrow \arctan a = \frac{3}{2} \Rightarrow a = \tan(3/2).
\]

4. Recall that the equation for the unit circle is \( x^2 + y^2 = 1 \), which means that the portion of the curve in the first quadrant may be described by the function \( f(x) = \sqrt{1 - x^2} \) for \( 0 \leq x \leq 1 \).
(a) Use geometry to show that
\[
\int_0^a \sqrt{1-x^2} \, dx = \frac{\pi}{4} - \frac{1}{2} \cos^{-1}(a) + \frac{a}{2} \sqrt{1-a^2} \quad (\text{for } 0 \leq a \leq 1).
\]

Hint: Let \( \theta \) be such that \( \cos(\theta) = a \). Recall that the area of a sector of a circle with angle \( \theta \) is \( \frac{\theta}{2} \). Consider the images below.

\[ \text{Solution: Area of the first region on the right side in the picture above is the area of one quarter of the unit circle, which is equal to } \frac{\pi}{4}. \]

And area of the region in the middle picture on the right side is the area of the sector with angle \( \theta \) which is \( \theta/2 \). And lastly, the area of the triangle in the rightmost figure is \( \frac{1}{2} \times \text{base} \times \text{height} \) with base = \( a \) and height = \( \sqrt{1-a^2} \).

Therefore, the area of the desired region on the left hand side
\[
\int_0^a \sqrt{1-x^2} \, dx = \frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{2} \times a \times \sqrt{1-a^2} \]
\[
= \frac{\pi}{4} - \frac{\arccos a}{2} + \frac{1}{2} \times a \times \sqrt{1-a^2} \quad (\text{since } \cos \theta = a \Rightarrow \theta = \arccos a)
\]

(b) Now show that
\[
F(x) = -\frac{1}{2} \cos^{-1} x + \frac{x}{2} \sqrt{1-x^2}
\]
is an anti-derivative to \( \sqrt{1-x^2} \).

\[ \text{Solution: } F'(x) = \frac{1}{2\sqrt{1-x^2}} + \frac{1}{2} \sqrt{1-x^2} - \frac{x^2}{2\sqrt{1-x^2}} = \sqrt{1-x^2}. \]

Or because by the definition of the anti-derivative and the previous part, one can say that the above is the anti-derivative of \( \sqrt{1-x^2} \).

(c) Explain why the term \( \frac{\pi}{4} \) is not needed from the integral calculation earlier.

\[ \text{Solution: Since any two anti-derivatives of a function differ by a constant, the term } \frac{\pi}{4} \text{ is not needed here.} \]