Instructions: Put the first and last name of everyone in your group at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

1. For each of the limits in this problem, complete the following items.

   (i) Express the limit as a definite integral over the given interval.
   (ii) Use geometry to evaluate the integral and hence also evaluate the limit.

(a) \( \lim_{n \to \infty} \sum_{i=1}^{n} (4x^*_i - 1) \Delta x \) over \([2, 5]\).

   Solution: The above limit equals \( \int_{2}^{5} (4x - 1) \, dx \).

Let \( f(x) = 4x - 1 \), which represents a line. By drawing its graph (as shown in the picture), we notice that the integral is the sum of the areas of the rectangle with vertices \((2, 0), (2, f(2)), (5, 0)(5, f(2))\) and the triangle with vertices \((2, f(2)), (5, f(5)), (5, f(2))\). And \( f(2) = 7, f(5) = 19 \). Therefore the area of the rectangle is \( 3 \times 7 = 21 \). The area of the triangle is \( \frac{1}{2} \times 3 \times 12 = 18 \). Hence,

\[
\lim_{n \to \infty} \sum_{i=1}^{n} (4x^*_i - 1) \Delta x = \int_{2}^{5} f(x) \, dx = 21 + 18 = 39.
\]

(b) \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 3 + \sqrt{4 - (x^*_i - 4)^2} \right) \Delta x \) over \([2, 4]\).

   Solution: This limit equals \( \int_{2}^{4} (3 + \sqrt{4 - (x - 4)^2}) \, dt \).

The function \( y = 3 + \sqrt{4 - (x - 4)^2} \) over \([2, 4]\) represents the upper left half of the circle (of radius 2)

\[
(y - 3)^2 = 4 - (x - 4)^2 \quad \text{i.e.} \quad (x - 4)^2 + (y - 3)^2 = 4.
\]
Therefore, area of this is one fourth of this circle, which is given by $\frac{\pi}{4} \times 4 = 1$.

Therefore,

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left( 3 + \sqrt{4 - (x_i^* - 4)^2} \right) \Delta x \text{ over } [2, 4] = \int_{2}^{4} (3 + \sqrt{4 - (x - 4)^2}) \, dt = 1$$

(c) $\lim_{n \to \infty} \sum_{i=1}^{n} |x_i^* - 3| \Delta x \text{ over } [0, 4]$.

**Solution:** This limit equals $\int_{0}^{4} |x - 3| \, dx$

Geometrically, this represents the area of the region under the graph of the function $f(x) = |x - 3|$ for $x \in [0, 4]$; and this is union of two triangles with vertices $(0, f(0)), (3, f(3)), (0, 0)$ and $(3, f(3)), (4, 0), (4, f(4))$. The areas of the two triangles is

$$\frac{1}{2} \times 3 \times 3 = \frac{9}{2} \text{ and } \frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$ 

Therefore,

$$\lim_{n \to \infty} \sum_{i=1}^{n} |x_i^* - 3| \Delta x \text{ over } [0, 4] = \int_{0}^{4} |x - 3| \, dx = \frac{9}{2} + \frac{1}{2} = 5$$

2. Suppose you know the values of the following definite integrals.

$$\int_{-1}^{7} f(x) \, dx = 10 \quad \int_{-1}^{5} f(x) \, dx = -2$$

$$\int_{-1}^{7} g(x) \, dx = 6 \quad \int_{-1}^{5} g(x) \, dx = 1$$

Use the properties of the definite integral in order to evaluate the following integrals.

(a) $\int_{-1}^{7} (f(x) - 4g(x)) \, dx$

**Solution:**

$$\int_{-1}^{7} (f(x) - 4g(x)) \, dx = \int_{-1}^{7} f(x) \, dx - 4 \int_{-1}^{7} g(x) \, dx = 10 - 4 \times 6 = -14.$$ 

(b) $\int_{-1}^{7} g(x) \, dx$

**Solution:**

$$\int_{-1}^{7} g(x) \, dx = - \int_{5}^{1} g(x) \, dx = -1.$$
(c) $\int_{5}^{7} f(x) \, dx$

Solution:

$$\int_{5}^{7} f(x) \, dx = \int_{-1}^{7} f(x) \, dx - \int_{-1}^{5} f(x) \, dx = 10 - (-2) = 12.$$ 

(d) $\int_{5}^{7} (5f(x) - 6g(x) + 2) \, dx$

Solution:

$$\int_{5}^{7} (5f(x) - 6g(x) + 2) \, dx = 5 \int_{5}^{7} f(x) \, dx - 6 \int_{5}^{7} g(x) \, dx + \int_{5}^{7} 2 \, dx$$

$$= 5 \times 12 - 6 \int_{-1}^{7} g(x) \, dx + 6 \int_{-1}^{5} g(x) \, dx + 2 \times 2$$

$$= 60 - 36 + 6 + 4 = 34$$

(e) $\int_{-1}^{7} 2h(x) \, dx$ if $h(x) \geq f(x)$ for all $x$

Solution:

$$\int_{-1}^{7} 2h(x) \, dx \geq \int_{-1}^{7} 2f(x) \, dx = 20$$

3. (a) Use geometry to evaluate the three integrals below.

(i) $\int_{0}^{6} 4 \, dx$  
(ii) $\int_{0}^{6} x \, dx$  
(iii) $\int_{0}^{6} 4x \, dx$

Solution: (i) $\int_{0}^{6} 4 \, dx$ is the area of the rectangle with sides 4 and 6. Therefore $\int_{0}^{6} 4 \, dx = 24$

(ii) $\int_{0}^{6} x \, dx$ denotes the area of the triangle with vertices $(0,0), (6,6), (6,0)$. Therefore $\int_{0}^{6} x \, dx = \frac{1}{2} \times 6 \times 6 = 18$

(iii) $\int_{0}^{6} 4x \, dx$ denotes the area of the triangle with vertices $(0,0), (6,24), (6,0)$. Therefore $\int_{0}^{6} 4x \, dx = \frac{1}{2} \times 6 \times 24 = 72$

(b) Do you agree or disagree with the line below? Explain.

$$\int_{a}^{b} f(x)g(x) \, dx = \left( \int_{a}^{b} f(x) \, dx \right) \left( \int_{a}^{b} g(x) \, dx \right)$$
Solution: Disagree! For example, take $a = 1, b = 2$ and $f(x) = 1, g(x) = 2$, then

\[
\int_a^b f(x)g(x) \, dx = \int_1^2 2 \, dx = 2 \times 1 = 2
\]

\[
\int_a^b f(x) \, dx \int_a^b g(x) \, dx = \int_1^2 1 \, dx \int_1^2 2 \, dx = 1 \times 1 + 2 \times 1 = 3
\]

Therefore, $\int_a^b f(x)g(x) \, dx$ is not necessarily always equal to $\int_a^b f(x) \, dx \int_a^b g(x) \, dx$.

4. For each of the integrals in this problem, complete the following items.

(i) Sketch the area represented by the definite integral.

(ii) Approximate the area with $R_4$ (sum with four subintervals & right endpoints). (You can check your answer with the demonstration!)

(iii) Compute the value of the integral using the limit definition of the definite integral.

You will find the following formulas helpful for part (iii).

\[
\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 + 1 = n. \tag{1}
\]

\[
\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}. \tag{2}
\]

\[
\sum_{i=1}^{n} i^2 = 1 + 4 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3}
\]

\[
\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}. \tag{4}
\]

(a) $\int_{4}^{7} (x+8) \, dx$

Solution: (i)

(b) Let $f(x) = x + 8$. For $R_4$, $\Delta x = \frac{7-4}{4} = \frac{3}{4}$. And $x_i = 4 + i\Delta x = 4 + \frac{3i}{4}$. Therefore,

\[
R_4 = \Delta x (f(x_1) + f(x_2) + f(x_3) + f(x_4))
\]

\[
= \frac{3}{4} \left( \left( 4 + \frac{3}{4} + 8 \right) + \left( 4 + \frac{6}{4} + 8 \right) + \left( 4 + \frac{9}{4} + 8 \right) + \left( 4 + \frac{12}{4} + 8 \right) \right).
\]
(iii) \( f(x) = x + 8, \Delta x = \frac{7-i}{n} = \frac{3}{n} \). \( x_i = 4 + \frac{3i}{n} \).

\[
R_n = \sum_{i=1}^{n} \Delta x f(x_i) = \sum_{i=1}^{n} \frac{3}{n} \left( 4 + \frac{3i}{n} + 8 \right) = \frac{3}{n} \sum_{i=1}^{n} \left( 12 + \frac{3i}{n} \right)
\]

\[
= \frac{3}{n} \sum_{i=1}^{n} 12 + \frac{3}{n} \sum_{i=1}^{n} \frac{3i}{n} = \frac{3}{n} \times 12 \sum_{i=1}^{n} 1 + \frac{9}{n^2} \sum_{i=1}^{n} i
\]

\[
= \frac{36n}{n} + \frac{9}{n^2} \frac{n(n+1)}{2} = 36 + \frac{9}{2} \times \frac{n+1}{n}
\]

Therefore

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( 36 + \frac{9}{2} \times \frac{n+1}{n} \right) = 36 + \frac{9}{2}.
\]

(b) \( \int_{-1}^{3} x^2 \, dx \)

Solution: (i)

(ii) Let \( f(x) = x^2 \). For \( R_4, \Delta x = \frac{3-(-1)}{4} = 1 \). And \( x_i = -1 + i \Delta x = -1 + i \).

Therefore,

\[
R_4 = \Delta x (f(x_1) + f(x_2) + f(x_3) + f(x_4))
\]

\[
= (-1 + 1)^2 + (-1 + 2)^2 + (-1 + 3)^2 + (-1 + 4)^2 = 0 + 1 + 4 + 9 = 14
\]

(iii) \( f(x) = x^2, \Delta x = \frac{3-(-1)}{n} = \frac{4}{n} \). \( x_i = -1 + \frac{4i}{n} \).

\[
R_n = \sum_{i=1}^{n} \Delta x f(x_i) = \sum_{i=1}^{n} \frac{4}{n} \left( -1 + \frac{4i}{n} \right)^2 = \frac{4}{n} \sum_{i=1}^{n} \left( 1 + \frac{16i^2}{n^2} - \frac{8i}{n} \right)
\]

\[
= \frac{4}{n} \sum_{i=1}^{n} 1 + \frac{64}{n^3} \sum_{i=1}^{n} i^2 - \frac{32}{n^2} \sum_{i=1}^{n} i
\]

\[
= \frac{4n}{n} \times n + \frac{64}{n^3} \times n \frac{(n+1)(2n+1)}{6} - \frac{32}{n^2} \times \frac{n(n+1)}{2}
\]

\[
= 4 + \frac{64}{6} \times \frac{(n+1)(2n+1)}{n^2} - 16 \times \frac{n+1}{n}
\]

Therefore

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( 4 + \frac{64}{6} \times \frac{(n+1)(2n+1)}{n^2} - 16 \times \frac{n+1}{n} \right) = 4 + \frac{64}{6} \times 2 - 16 = \frac{28}{3}.
\]
5. We have that \( \int_a^b f(x) dx \) represents the net area beneath the curve \( f(x) \) between \( a \) and \( b \) for \( a < b \). Show that \( \int_b^a f(x) dx = -\int_a^b f(x) \) must be true if you want

\[
\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx
\]

to be true for all \( a, b, \) and \( c \) independent of order.

**Solution:** Suppose

\[
\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx
\]

for any \( a,b,c \). Then, in particular, it also holds for \( c = a \). This gives

\[
\int_a^b f(x) dx + \int_b^a f(x) dx = \int_a^a f(x) dx = 0
\]

i.e. \( \int_a^b f(x) dx + \int_b^a f(x) dx = 0 \)

i.e. \( \int_a^b f(x) dx = -\int_b^a f(x) dx \),

which is what we wanted to show.