Some questions on this worksheet reference a computer-generated demonstration (access it through Moodle) to aid in conceptual understanding of the questions.

1. Consider the function \( f(x) = x^2 - 4x + 5 \) over the interval \([0, 2]\). Let \( A = \int_0^2 f(x) \, dx \).

   (a) Show that \( f \) is decreasing over \((0, 2)\).

   Solution: \( f'(x) = 2x - 4 < 0 \) since \( x < 2 \). Therefore, \( f(x) \) is decreasing on \([0, 2]\).

   (b) Will the sum created from using LEFT endpoints \((L_n)\) be an overestimate (larger than the actual value) or an underestimate (smaller than the actual value) over the interval \((0, 2)\)? Will the sum created from using RIGHT endpoints \((R_n)\) be an overestimate or an underestimate? Explain. Rank in order of smallest to largest \( A, L_n, \) and \( R_n \).

   Solution: Since, \( f(x) \) is decreasing on \([0, 2]\), the left endpoints \( L_n \) will be an overestimate and \( R_n \) will be underestimate of the actual area. So, rank-wise: \( L_n \geq A \geq R_n \).

   (c) Suppose you split the interval \([0, 2]\) into \( n \) subintervals of equal length \((\Delta x = \frac{2}{n})\). Graphically demonstrate how the error from either \( L_n \) or \( R_n \) is less than \((f(0) - f(2))\Delta x = \frac{8}{n}\). What happens to the error from either \( L_n \) or \( R_n \) as we divide the interval into more and more subintervals? How do \( L_n \) and \( R_n \) compare to \( A \) as \( n \to \infty \)?

   Solution: From moodle, one can see that the error from the \( j \)th rectangle for \( R_n \) (or for \( L_n \)) is at most the area of the smaller rectangle with sides \( f(x_{j-1}) - f(x_j) \) and \( \Delta x \). Therefore, the total error would be

   \[
   \sum_{j=1}^{n} \Delta x(f(x_{j-1}) - f(x_j)) = \Delta x [(f(x_0) - f(x_1)) + (f(x_1) - f(x_2)) + \ldots + (f(x_{n-1}) - f(x_n))] \\
   = \Delta x (f(x_0) - f(x_n)) = \Delta x (f(0) - f(2)) = \frac{2}{n} (5 - 1) = \frac{8}{n}.
   \]

   The error term \( \frac{8}{n} \) keeps becoming smaller and smaller as we increase the number \( n \) of the subintervals. In fact, it goes to zero as \( n \) approaches infinity. Therefore, \( L_n \) or \( R_n \) approaches the area \( A \).

   (d) Suppose that you have a function \( g(x) \) that is increasing over the interval \([a, b]\), will \( L_n \) or \( R_n \) provide an underestimate? Which will provide an overestimate? What is the maximum amount that either approximation will differ from \( \int_a^b g(x) \, dx \)? What will happen to the error as the interval is divided into more and more subintervals?

   Solution: Since \( g(x) \) is increasing on \([a, b]\), the left endpoints \( L_n \) will be an
underestimate and $R_n$ will be overestimate of the actual area. So, rank-wise: $R_n \geq A \geq L_n$. The error from the $j$th rectangle for $R_n$ (or for $L_n$) is at most the area of the smaller rectangle with sides $g(j) - g(j-1)$ and $\Delta x$. Therefore, the total error would be

$$\sum_{j=1}^{n} \Delta x(g(x_j) - g(x_{j-1})) = \Delta x[(g(x_1) - g(x_0)) + (g(x_2) - g(x_1)) + \ldots + (g(x_n) - g(x_{n-1}))]$$

$$= \Delta x(g(x_n) - g(x_0)) = \Delta x(g(b) - g(a)) = \frac{b-a}{n}(g(b) - g(a)).$$

The error term $\frac{b-a}{n}(g(b) - g(a))$ keeps decreasing as we increase the number $n$ of the subintervals, since the numerator $(b - a)(g(b) - g(a))$ is fixed as $a$ and $b$ are not changing. In fact, it goes to zero as $n$ approaches infinity. Therefore, $L_n$ or $R_n$ approaches the area $A$.

2. Suppose that the speed of a runner was always increasing for $10 \leq t \leq 30$. Use the data on the runner’s speed to give an upper and a lower estimate on the distance traveled from 10 seconds to 30 seconds.

<table>
<thead>
<tr>
<th>time (seconds)</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>24</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed (ft/sec)</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>9.5</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution:** As in Question 1(d), we note that $R_n$ will be overestimate and $L_n$ will be underestimate since the runner’s speed is always increasing for $t \in [10, 30]$. In this case, we have $n = 4$ and $\Delta x$ is not fixed.

$$R_n = (12 - 10) \times 8 + (20 - 12) \times 9 + (24 - 20) \times 9.5 + (30 - 24) \times 10 = 186,$$

and

$$L_n = (12 - 10) \times 7 + (20 - 12) \times 8 + (24 - 20) \times 9 + (30 - 24) \times 9.5 = 171.$$

3. For each of the integrals in this problem, complete the following items.

(i) Sketch the area represented by the definite integral.

(ii) Compute the area represented using geometry.

(iii) Approximate the area with $R_4$ (sum with four subintervals & right endpoints).

(You can check your answer with the demonstration!)

(iv) Compute the value of the integral using the limit definition of the definite integral.
You will find the following formulas helpful for part (iv).

\[\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 + 1 = n.\] (1)

\[\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.\] (2)

\[\sum_{i=1}^{n} i^2 = 1 + 4 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.\] (3)

\[\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.\] (4)

(a) \( \int_{2}^{8} 2x \, dx \)

**Solution:**

(i) \( f(x) = 2x \)

(ii) The area of the desired (colored) region is the sum of the areas of a triangle with base 6 height \( f(8) - f(2) = 12 \) and a rectangle with sides 6 and \( f(2) \). Therefore the total area is

\[ \frac{1}{2} \times 6 \times 12 + 6 \times 4 = 60. \]

(iii) For \( R_4 \), \( \Delta x = \frac{8-2}{4} = \frac{3}{2} \). And for \( i = 1, 2, 3, 4; x_i = 2 + \frac{3i}{2} \). This gives

\[ R_4 = \Delta x \sum_{i=1}^{4} f(x_i) = \frac{3}{2} (f(2+3/2) + f(2+6/2) + f(2+9/2) + f(2+12/2)) \]

\[ = \frac{3}{2} (2(2+3/2) + 2(2+6/2) + 2(2+9/2) + 2(2+12/2)) \]

(iv) For \( R_n \), \( \Delta x = \frac{8-2}{n} = \frac{6}{n} \). And for \( i = 1, 2, \ldots, n; x_i = 2 + \frac{6i}{n} \). This gives

\[ R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{6}{n} \sum_{i=1}^{n} f \left( 2 + \frac{6i}{n} \right) \]

\[ = \frac{6}{n} \sum_{i=1}^{n} \left( 2 + \frac{6i}{n} \right) = \frac{12}{n} \sum_{i=1}^{n} \left( 2 + \frac{6i}{n} \right) = \frac{12}{n} \sum_{i=1}^{n} 2 + \frac{12}{n} \sum_{i=1}^{n} \frac{6i}{n} \]

\[ = \frac{24}{n} \sum_{i=1}^{n} \frac{1 + 72}{n^2} \sum_{i=1}^{n} i = \frac{24}{n} \times n + \frac{72}{n^2} \times \frac{n(n+1)}{2} = 24 + 36 \times \frac{n+1}{n} \]
Therefore,
\[ \int_2^8 2x \, dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( 24 + 36 \cdot \frac{n+1}{n} \right) = 24 + 36. \]

(b) \[\int_0^4 (5x + 3) \, dx\]

**Solution:** (i) \( f(x) = 5x + 3 \)

(ii) The area of the desired (colored) region can be found as in the previous part. But, one can also look at it as a trapezoid with sides \( f(0) \) and \( f(4) \) and height 4. And, its area is given by
\[ \frac{1}{2} \times 4 \times (f(0) + f(4)) = 2 \times (3 + 23) = 52. \]

(iii) For \( R_4 \), \( \Delta x = \frac{4-0}{4} = 1 \). And for \( i = 1, 2, 3, 4; \) \( x_i = 0 + i \Delta x = i \). This gives
\[ R_4 = \Delta x \sum_{i=1}^{4} f(x_i) = f(1) + f(2) + f(3) + f(4) = 8 + 13 + 18 + 23 \]

(iv) For \( R_n \), \( \Delta x = \frac{4-0}{n} = \frac{4}{n} \). And for \( i = 1, 2, \ldots, n; \) \( x_i = 0 + \frac{4i}{n} \). This gives
\[ R_n = \Delta x \sum_{i=1}^{n} f(x_i) = \frac{4}{n} \sum_{i=1}^{n} f \left( \frac{4i}{n} \right) = \frac{4}{n} \sum_{i=1}^{n} \left( 5 \times \frac{4i}{n} + 3 \right) = \frac{4}{n} \sum_{i=1}^{n} \left( \frac{20}{n} \times i + 3 \right) = \frac{80}{n^2} \sum_{i=1}^{n} i + \frac{12}{n} \sum_{i=1}^{n} 1 \]
\[ = \frac{80}{n^2} \times \frac{n(n+1)}{2} + \frac{12}{n} \times n = 40 \times \frac{n+1}{n} + 12 \]

Therefore,
\[ \int_0^4 5x + 3 \, dx = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( 12 + 40 \times \frac{n+1}{n} \right) = 12 + 40. \]

4. Let \( f(x) \) be a positive, continuous function on \([a, b]\). The area beneath the curve over this integral may be represented by the definite integral \( \int_a^b f(x) \, dx \). Provide the limit that computes the area beneath \( f \) if you use
(a) rectangles generated by right endpoints.
Solution: Here $\Delta x = \frac{b-a}{n}$ and for $i = 1, 2, \ldots, n$; $x_i = a + i \Delta x = a + i \frac{b-a}{n}$. Then,

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i).$$

(b) rectangles generated by left endpoints.
Solution: Here $\Delta x = \frac{b-a}{n}$.
For $i = 0, 1, \ldots, n-1$; if $x_i = a + i \Delta x = a + i \frac{b-a}{n}$. Then,

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \Delta x \sum_{i=0}^{n-1} f(x_i).$$

Or for $i = 1, 2, \ldots, n$; if $x_i = a + (i - 1) \frac{b-a}{n}$,

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i).$$

(c) rectangles generated by midpoints.
Solution: Here $\Delta x = \frac{b-a}{n}$ and for $i = 1, 2, \ldots, n$; let $x_i = a + i \Delta x = a + i \frac{b-a}{n}$, and

$$x_i^* = \frac{x_i + x_{i+1}}{2}.$$  

Then,

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i^*).$$

5. The following limit represents the area beneath some curve. The rectangles were generated by right endpoints.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{25 - \left( \frac{5i}{n} \right)^2} \cdot \frac{5}{n}$$

(a) Sketch a region whose area is represented by the above limit.
Solution: (i) The shaded region in the picture below.

(b) Write down the definite integral that this limit equals.
Solution:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{25 - \left( \frac{5i}{n} \right)^2} \cdot \frac{5}{n} = \int_0^5 \sqrt{25 - x^2} \, dx.$$
(c) Determine the value of the limit by computing the represented area geometrically.

**Solution:** Since the function \( y = \sqrt{25 - x^2} \) represents the upper half of the circle \( x^2 + y^2 = 25 \), as shown in Part (a), the integral is the area of a quarter of the circles and equals

\[
\frac{1}{4} \times 25\pi
\]