Worksheet #16
Math 221

Instructions: Put the first and last name of everyone in your group at the top of your paper. Everyone is to do their own worksheet but only one from each group is graded with the score shared. Be sure to show your work and explain your reasoning.

Some questions on this worksheet reference a computer-generated demonstration (access it through Moodle) to aid in conceptual understanding of the questions.

1. In this problem, determine the largest rectangle that fits inside the first quadrant and beneath the curve \( f(x) = \sqrt{9-x^2} = 3\sqrt{1-\left(\frac{x}{3}\right)^2} \).

   (a) Provide a sketch of \( f(x) \) as it appears in the first quadrant. Label your graph!

   Answer:

   \[
   \begin{align*}
   &\text{(b) If one corner of an inscribed rectangle is at the point (0, 0) and the opposite corner is on the curve } f(x), \text{ what is the area of the rectangle with base length } x? \\
   &\text{Answer:} \\
   \end{align*}
   \]

   \[
   \begin{align*}
   &A = x \cdot f(x) \\
   \end{align*}
   \]  

   (c) Based on the demonstration, what is the approximate length of the base of the largest rectangle?

   Answer: It is about x=2.12

   (d) Use calculus to determine the exact dimensions of the largest inscribed rectangle. Compare a numerical approximation of your value to the approximation you found in the previous step.

   Answer:

   \[
   \begin{align*}
   &A = xf(x) = x(\sqrt{9-x^2}) \to A'(x) = \sqrt{9-x^2} + x \left( \frac{-x}{\sqrt{9-x^2}} \right) \\
   \end{align*}
   \]  

   Setting this equation equal to zero and solving for the x-value, we find that the exact value is \( x = \sqrt{\frac{9}{2}} \). This is approximately 2.12.
(e) Building from your responses above, what are the dimensions of the largest rectangle that may be inscribed inside a circle of radius 3?

**Answer:** The inscribed rectangle is largest when it is a square, as can be seen from the equation. If we plug in our x-value to the function, we get the same value back, so the dimensions of the square are $2\sqrt{\frac{9}{2}}$ by $2\sqrt{\frac{9}{2}}$.

2. In this problem, determine the largest rectangle that fits inside the ellipse $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$ by first considering the largest rectangle in the first quadrant beneath the curve $f(x) = \sqrt{9 - \frac{9x^2}{25}} = 3\sqrt{1 - \left(\frac{x}{5}\right)^2}$.

(a) Based on the demonstration, what is the approximate length of the base of the largest rectangle?

**Answer:** It looks like the largest rectangle has base 3.51.

(b) Use calculus to determine the exact dimensions of the largest inscribed rectangle. Compare a numerical approximation of your value to the approximation you found in the previous step.

**Answer:**

\[
A = xf(x) = x \left(3\sqrt{1 - \frac{x^2}{25}}\right) \rightarrow A'(x) = \sqrt{1 - \frac{x^2}{25}} + x \left(\frac{-3x}{25\sqrt{1 - \frac{x^2}{25}}}\right) \tag{3}
\]

Setting this equal to zero and solving for the x-value, we get a value of $x = \sqrt{\frac{25}{2}}$. This is approximately equal to 3.53.

(c) Building from your responses above, what are the dimensions of the largest rectangle that may be inscribed inside an ellipse with equation $\left(\frac{x}{5}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$?

**Answer:** By plugging in this value to the function, we get that the dimensions of the largest rectangle are $2\sqrt{\frac{25}{2}}$ by $2\sqrt{\frac{9}{2}}$.

(d) Conjecture the dimensions of the largest rectangle that may be inscribed inside an ellipse with equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

**Answer:** By doing the same process, we can find the general form of the x-value. We do this by solving the general derivative.

\[
A'(x) = \sqrt{b^2 - \frac{b^2 x^2}{a^2}} - x \left(\frac{xb^2}{a^2\sqrt{b^2 - \frac{b^2 x^2}{a^2}}}\right) \tag{4}
\]

Setting this equal to zero and solving, we get the general base of the rectangle will be $x = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}$. Therefore, the height will be $f(x) = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}}$. 

2
3. Determine the right cylinder of largest volume that fits inside a sphere of radius $R$ by completing the steps below. Let $R$ represent the radius of the sphere (a fixed value), $r$ represent the radius of the cylinder (variable).

(a) Come up with a formula for the height $H$ of the cylinder in terms of $R$ and $r$.

**Answer:** By making a right triangle with the radius of the sphere and the radius of the cylinder in a plane, we can find the height using the Pythagorean formula. This is $H = 2\sqrt{R^2 - r^2}$.

(b) What is the formula for the volume $V$ of the cylinder in terms of $R$ and $r$?

**Answer:**

$$V = \pi r^2 h \Rightarrow V = 2\pi r^2 \sqrt{R^2 - r^2} \quad (5)$$

(c) Use calculus to determine the radius $r$ that leads to the largest cylinder by volume that fits inside a sphere of radius $R$.

**Answer:**

$$\frac{dV}{dr} = 2\pi \left(2r\sqrt{R^2 - r^2} + r^2 \left(\frac{-r}{\sqrt{R^2 - r^2}}\right)\right)$$

$$= 2\pi \left(\frac{2r(R^2 - r^2) - r^3}{\sqrt{R^2 - r^2}}\right)$$

$$= 2\pi \left(\frac{r(2R^2 - 3r^2)}{\sqrt{R^2 - r^2}}\right)$$

We solve for critical points by setting this equation equal to zero. We obtain $r = 0$ or $r = \pm \sqrt{\frac{2}{3}}R$. We can safely toss out the negative critical value because we’re only considering $0 \leq r \leq R$. Clearly, $r = 0$ (an endpoint) leads to no volume (a minimum), $V(R) = 0$, but $V\left(\sqrt{\frac{2}{3}}R\right)$ is the maximum (by first derivative test).

We get that the maximum volume of the cylinder occurs at $r = \sqrt{\frac{2R^2}{3}}$.

(d) Suppose $R = 1$. According to the demo, approximately what radius $r$ leads to the largest cylinder by volume? Compare this answer to the exact answer obtained in your response to the previous part.

**Answer:** The approximate value from the demonstration is $r = .824$. Our method gives a value of $r = \sqrt{\frac{2}{3}} \approx 0.816$.

(e) If you took the largest inscribed rectangle within a circle and spun the 2D object about the $y$-axis, the resulting shape is a cylinder inside a sphere, but this is not the largest cylinder by volume. Explain why this is the case.

**Answer:** Maximizing a rectangle in a circle gives a square. However, when you maximize a cylinder, the volume increases with the square of the radius and...
linearly with the height. Therefore, the weighting of the variables is different in
the two different cases.

4. A 3 ft tall TV is hung on a wall so that the lower edge is 4 ft above eye level. How far
from the wall should someone sit to get a maximum viewing angle \( \theta \)?
(Hint: as shown in the demonstration, \( \theta = \beta - \alpha \). In order to maximize \( \theta \), you will
first need to determine formulas for \( \alpha \) and \( \beta \) that use the \text{arctan} function so you can
relate \( \theta \) back to \( x \), the distance a person should sit from the wall.)

**Answer:** Define three angles. \( \alpha \) is the angle from the viewer to the bottom of the
tv and to the ground, \( \beta \) is the angle from the viewer to the top of the tv and to the
ground, and \( \theta \) is the angle from the viewer to top of the tv and to the bottom of the
tv. Therefore, \( \alpha + \theta = \beta \). We seek to maximize the viewing angle \( \theta \), so we rewrite this
equation and find relations for the other two angles.

\[
\beta - \alpha = \theta, \quad \tan \beta = \frac{7}{x}, \quad \tan \alpha = \frac{4}{x}.
\]

(6)

We now find the derivative in order to maximize the viewing angle with respect to the
distance away from the TV.

\[
\frac{d\theta}{dx} = \frac{d}{dx} \left( \text{arctan} \left( \frac{7}{x} \right) - \text{arctan} \left( \frac{4}{x} \right) \right) = \frac{-7}{x^2 + 49} + \frac{4}{x^2 + 16}
\]

(7)

Setting this equation equal to zero and finding the \( x \) value, we get

\[
7(x^2 + 16) = 4(x^2 + 49) \Rightarrow 3x^2 = 4 \cdot 49 - 7 \cdot 16 = 4(49 - 28) = 4(21) = 84 \Rightarrow x^2 = 28.
\]

5. Determine the point on the line \( y = 3x - 5 \) that is closest to the point \((-7, 15)\).

**Answer:** We seek to minimize the distance formula, but that will happen at the same
time as the minimum of the distance squared.

\[
z = \sqrt{(x + 7)^2 + (3x - 20)^2}
\]

\[
z^2 = (x + 7)^2 + (3x - 20)^2
\]

Differentiating, we have

\[
2z \frac{dz}{dx} = 2(x + 7) + 2(3x - 20) \cdot 3
\]

\[
\frac{dz}{dx} = \frac{x + 7 + 9x - 60}{z} = \frac{10x - 53}{z}.
\]

Setting this equal to zero, we obtain \( x = \frac{53}{10} \). We may use the First Derivative Test to
verify that this is the \( x \)-coordinate for the closest point on the line.