Worksheet #9
Math 221

1. The lemniscate shown in Figure 1 is described by the equation

\[(x^2 + y^2)^2 = 4(x^2 - y^2)\]

and is used in typography to denote infinity (\(\infty\)).

(a) Find \(\frac{dy}{dx}\) for points on the graph.

\textbf{Sol.} Differentiating both sided w.r.t. \(x\) gives

\[2(x^2 + y^2)(2x + 2yy') = 4(2x - 2yy')\]
\[y'(4y(x^2 + y^2) + 8y) = 8x - 4x(x^2 + y^2)\]
\[\frac{dy}{dx} = y' = \frac{8x - 4x(x^2 + y^2)}{4y(x^2 + y^2) + 8y}\]

(b) Are there any points on the curve for which \(\frac{dy}{dx}\) is undefined?

\textbf{Ans.} From above, the denominator in \(dy/dx\) is 0 only if \(y = 0\). For \(y = 0\), from the equation \((x^2 + y^2)^2 = 4(x^2 - y^2)\) of the lemniscate, we have

\[x^4 = 4x^2 \Rightarrow x^2(x^2 - 4) = 0 \text{ or } x = 0, \pm 2\]

Thus, \(\frac{dy}{dx}\) is not defined at points \((\pm 2, 0)\) and \((0, 0)\).

(c) Determine the equation for the tangent line at the point \((\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}})\).

\textbf{Solution:} Slope of the tangent line at \((\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}})\) is

\[y'\left(\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}\right) = \frac{8\sqrt{3/2} - 4\sqrt{3/2}(3/2 + 1/2)}{4\sqrt{2}(3/2 + 1/2) + 8\sqrt{2}} = 0\]

The equation of the line is

\[y - \frac{1}{\sqrt{2}} = 0 \left(x - \sqrt{\frac{3}{2}}\right) = 0\]

or simply \(y = \frac{1}{\sqrt{2}}\).

2. Consider the relationship \(\sin(y) - \cos(y) = x^2\). A graph of this relationship for \(-3 \leq x \leq 3\) and \(-1 \leq y \leq 5\) is provided in Figure 2.
(a) Determine a formula for $\frac{dy}{dx}$.
Sol. $\cos(y) y' + \sin(y) y' = 2x$. Therefore,
$$\frac{dy}{dx} = y' = \frac{2x}{\cos(y) + \sin(y)}.$$ 

(b) Determine the highest and lowest points on the curve shown in Figure 2.
Sol. At the highest and the lowest points, the slope of the tangents is zero. For $-3 \leq x \leq 3$ and $-1 \leq y \leq 5$,
$$\frac{dy}{dx} = \frac{2x}{\cos(y) + \sin(y)} = 0 \Rightarrow x = 0.$$ 
This gives $\sin(y) - \cos(y) = 0 \Rightarrow y = \frac{\pi}{4}, \frac{5\pi}{4}$. So, the possible points are $(0, \pi/4)$ and $(0, 5\pi/4)$ and $(0, \pi/4)$ is the lowest while $(0, 5\pi/4)$ is the highest.

(c) Determine the points on the curve shown in Figure 2 with vertical tangent lines.
Sol. For the vertical tangent lines, the slope is infinite. The denominator $\cos y + \sin y$ in $dy/dx$ is zero only if $\cos y = -\sin y$ i.e.
$$\tan y = -1 \Rightarrow y = \frac{3\pi}{4}.$$ 
This implies $x^2 = \sin(3\pi/4) - \cos(3\pi/4)$ i.e. $x = \pm \sqrt{2}$. Now notice that at these points $(\pm \sqrt{2}, 3\pi/4)$, $dy/dx$ is actually infinite. So the points with vertical asymptotes are $(\pm \sqrt{2}, 3\pi/4)$.

(d) Identify all other points $(x, y)$ satisfying the relationship $\sin(y) - \cos(y) = x^2$ that have vertical tangent lines.
Sol. The only difference between this problem and the previous one is that here there are no bounds for $y$, and since $y = 3\pi/4 + k\pi/2$ for any integer will satisfy $\tan y = -1$ the solutions are $(\pm \sqrt{2}, 3\pi/4 + k\pi/2)$ for $k$ any integer.
3. Use implicit differentiation to determine \( y' \) if \( y = \arctan(x) \). (Hint: if \( y = \arctan(x) \), then \( \tan(y) = x \).)

**Sol.**

\[
y = \arctan(x) \Rightarrow \tan(y) = x
\]

Differentiating both sides w.r.t \( x \), we have

\[
\sec^2(y) y' = 1 \Rightarrow y' = \frac{1}{\sec^2(y)} = \frac{1}{1 + \tan^2(y)} = \frac{1}{1 + x^2}.
\]

4. A graph of \( y = x^x \) is shown in Figure 3.

(a) Determine \( \frac{dy}{dx} \).

**Sol.** Note that \( \ln y = \ln x^x \Rightarrow \ln y = x \ln x \). Therefore,

\[
\frac{1}{y} \frac{dy}{dx} = \ln x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1).
\]

(b) Give the formula for the tangent line to the curve at the point \((2, 4)\).

**Sol.** The slope of the tangent line at the point \((2, 4)\) is \( dy/dx \) at \( x = 2 \), i.e.

\[
2^2(\ln 2 + 1) = 4 \ln 2 + 4
\]

Thus the equation of the tangent line is given by

\[
y - 4 = (4 \ln 2 + 4)(x - 2).
\]

5. True or False? The tangent line to the function \( y = (x^2 + 1)^{\cos(x)} \) (shown in Figure 4) when \( x = 1 \) is horizontal. Justify your answer.

**Sol.** Note that \( \ln y = \ln(x^2 + 1)^{\cos(x)} = \cos(x) \ln(x^2 + 1) \). Therefore,

\[
\frac{1}{y} \frac{dy}{dx} = -\sin(x) \ln(x^2 + 1) + \cos(x) \frac{1}{x^2 + 1} 2x
\]

\[
\frac{dy}{dx} = y \left( -\sin(x) \ln(x^2 + 1) + \cos(x) \frac{1}{x^2 + 1} 2x \right)
\]

\[
= (x^2 + 1)^{\cos(x)} \left( -\sin(x) \ln(x^2 + 1) + \cos(x) \frac{1}{x^2 + 1} 2x \right)
\]
False! At $x = 1$, 
\[ \frac{dy}{dx} = 2^{\cos(1)}(-\sin(1)\ln 2 + \cos(1)\frac{1}{2}\ln 2) \neq 0 \]

6. Determine a formula for $\frac{dy}{dx}$ if $y = (f(x))^{g(x)}$ and $f(x) > 0$.

**Sol.** Note that $\ln y = \ln ((f(x))^{g(x)}) = g(x)\ln(f(x))$. Therefore,

\[
\frac{1}{y} \frac{dy}{dx} = g'(x)\ln(f(x)) + g(x)\frac{1}{f(x)} f'(x)
\]

\[
\frac{dy}{dx} = y \left( g'(x)\ln(f(x)) + g(x)\frac{1}{f(x)} f'(x) \right)
\]

\[
= (f(x))^{g(x)} \left( g'(x)\ln(f(x)) + g(x)\frac{1}{f(x)} f'(x) \right)
\]

Note that the condition $f(x) > 0$ is needed for $\ln(f(x))$ to make sense.