A WRITTEN SKETCH OF THE FUNDAMENTAL THEOREM OF CALCULUS

The Fundamental Theorem of Calculus states that if $f$ is continuous on $[a, b]$, then
\[
\frac{d}{dx} \left( \int_a^x f(t)dt \right) = f(x).
\]

Definition of Derivative.

By the definition of the derivative,
\[
\frac{d}{dx} \left( \int_a^x f(t)dt \right) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = \lim_{h \to 0} \frac{\int_a^x f(t)dt + \int_x^{x+h} f(t)dt - \int_a^x f(t)dt}{h}
\]
\[
= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)dt}{h}
\]
where the final equality holds since
\[
\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt.
\]

Extreme Value Theorem.

By the Extreme Value Theorem, the function $f$ has a maximum value at a point $M$ in $[x, x + h]$ and $f$ has a minimum value at a point $m$ in $[x, x + h]$ for a fixed $h > 0$. Thus, by the definition of the integral, we have that
\[
h \cdot f(m) \leq \int_x^{x+h} f(t)dt \leq h \cdot f(M).
\]

Algebra.

Now, when $h \neq 0$, we can divide each term in the previous inequality by $h$ giving
\[
f(m) \leq \frac{1}{h} \int_x^{x+h} f(t)dt \leq f(M).
\]
Note that we can think of $m$ and $M$ as functions of $h$ since, as $h$ increases or decreases, the interval which we are considering the max and min values gets larger or smaller respectively.
Intermediate Value Theorem. 

Since 

$$f(m) \leq \frac{1}{h} \int_{x}^{x+h} f(t)dt \leq f(M),$$

by the Intermediate Value Theorem, we have that there is a point $c$ in the interval $[x, x + h]$ such that 

$$f(c) = \frac{1}{h} \int_{x}^{x+h} f(t)dt$$

for each value of $h > 0$.

Limit of average values $c$.

As $h$ goes to 0 the $c$’s must go to $x$ since $c$ is between $x$ and $x+h$. Because $f$ is continuous, 

$$\lim_{c \to x} f(c) = f(x)$$

and so if we take the limit as $h$ goes to zero on both sides, we get 

$$f(x) = \lim_{h \to 0} f(c) = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t)dt =$$

$$\lim_{h \to 0} \frac{\int_{a}^{x+h} f(t)dt - \int_{a}^{x} f(t)dt}{h} = \frac{d}{dx} \left( \int_{a}^{x} f(t)dt \right).$$