1. 30 points (Essentially question 4-68). Daily water demand $D$ (in millions of gallons) in Phoenix is (approximately) distributed with mean 310 and standard deviation 50. City reservoirs hold 350 (again, in millions of gallons). normally
(a) 20 points What is the probability that demand exceeds reservoir capacity?
(b) 10 points Suppose that we want to increase the reservoir capacity so that the demand exceeds supply only 1% of the time. How large should the reservoirs be?

2. 20 points (Essentially Question 4-104 in text). The time between arrivals of aircraft at an airport is exponentially distributed with mean of 1 hour.
(a) 10 points What is the probability that more than 5 aircraft arrive within two hours.
(b) 10 points Suppose that the airport opens at 7 AM each morning. What is the probability that no aircraft arrive between 7 and 7:30 on Monday, Tuesday and Wednesday?

3. 120 points Let $U$ be a standard uniform random variable (i.e., a randomly-chosen number between 0 and 1). In other words, $U$ is a continuous random variable with density

$$f_U(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(a) 10 points Compute $E[U]$.
(b) 10 points Compute $E[U^2]$.
(c) 10 points Compute the variance of $U$.
(d) 10 points Compute $P\{U \leq -4\}$
(e) 10 points Compute $P\{U \leq .465\}$
(f) 10 points Compute $P\{U \leq 1.2\}$.
(g) 10 points Compute $F_U$, the cumulative distribution function of $U$.

Define $Y \overset{\text{def}}{=} 3U + 2$.
(h) 10 points Compute $P\{Y \leq 1.75\}$
(i) 10 points Compute $P\{Y \leq 4.65\}$
(j) 10 points  Compute $P\{Y \leq 6.4\}$.

(k) 10 points  Compute $F_Y$, the cumulative distribution function of $Y$.

(l) 10 points  Compute $f_Y$, the density of $Y$.

4. 30 points  Suppose that two continuous random variables have joint density

$$f_{X,Y}(s,t) = \begin{cases} \frac{1}{2}se^{-st} & \text{if } 0 \leq s \leq 2 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

(a) 10 points  Compute $f_X\left(\frac{3}{4}\right)$; i.e., the first marginal, evaluated at $\frac{3}{4}$.

(b) 10 points  Compute $f_X(5)$; i.e., the first marginal, evaluated at 5.

(c) 10 points  Compute $f_X$, the density of $X$. 

R. Sowers 2
Answers

1. Set $N \overset{\text{def}}{=} (D - 310)/50$; then $D = 310 + 50N$, and $N$ is a standard Gaussian.

   (a) 
   
   \[
   \Pr\{D \geq 350\} = \Pr\{310 + 50N \geq 350\} = \Pr\{N \geq 40/50\} = \Pr\{N \geq .8\} = 1 - \Pr\{N < 0.8\} = 1 - .788 = .212.
   \]

   (b) We want $\Pr\{D \geq x\} = .01$. Thus
   
   \[
   .01 = \Pr\{D \geq x\} = \Pr\{310 + 50N \geq x\}
   = \Pr\{N \geq (x - 310)/50\} = 1 - \Pr\{N < (x - 310)/50\}.
   \]
   
   We thus want $\Pr\{N < (x - 310)/50\} = 0.99$ so $(x - 310)/50 \approx 2.33$; i.e.,
   
   \[
   x = 310 + 2.33 \times 50 = 426.5.
   \]

2. (a) The number of aircraft $N$ arriving in two hours is Poisson with parameter 2.

   \[
   \Pr\{N > 5\} = 1 - \Pr\{N \leq 4\} = 1 - \sum_{j=0}^{4} e^{-2} \frac{2^j}{j!}.
   \]

   (b) The number of aircraft $N$ arriving in half an hour is Poisson with parameter 0.5; $N_M$, $N_T$, and $N_W$ are the number of aircraft arriving 7-7:30 on Monday, Tuesday, and Wednesday. They are independent.

   \[
   \Pr\{N_M = 0, N_T = 0, N_W = 0\} = (e^{-0.5})^3 = e^{-1.5}.
   \]

3. (a) $\mathbb{E}[U] = \int_{t=0}^{1} t \, dt = \frac{1}{2}$.

   (b) $\mathbb{E}[U^2] = \int_{t=0}^{1} t^2 \, dt = \frac{1}{3}$.

   (c) $\sigma_U^2 = \mathbb{E}[U^2] - [\mathbb{E}[U]]^2 = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

   (d) $\Pr\{U \leq -4\} = 0$

   (e) $\Pr\{U \leq .465\} = 0.465$.

   (f) $\Pr\{U \leq 1.2\} = 1$.

   (g)

   \[
   F_U(t) = \begin{cases} 
   0 & \text{if } t < 0 \\
   t & \text{if } 0 \leq t < 1 \\
   1 & \text{if } t \geq 1
   \end{cases}
   \]

   (h) $\Pr\{Y \leq 1.75\} = \Pr\{3U + 2 \leq 1.75\} = \Pr\{U \leq -.25/3\} = 0$.

   (i) $\Pr\{Y \leq 4.65\} = \Pr\{3U + 2 \leq 4.65\} = \Pr\{U \leq 2.65/3\} = \frac{2.65}{3}$. 

R. Sowers
(j) $\mathbb{P}\{Y \leq 6.4\} = \mathbb{P}\{3U + 2 \leq 6.4\} = \mathbb{P}\{U \leq 4.4/3\} = 1.$

(k) 
\[ F_Y(t) = \begin{cases} 
0 & \text{if } t < 2 \\
\frac{t-2}{3} & \text{if } 2 \leq t < 5 \\
1 & \text{if } t \geq 5 
\end{cases} \]

(l) 
\[ f_Y(t) = \begin{cases} 
\frac{1}{3} & \text{if } 2 < t < 5 \\
0 & \text{else} 
\end{cases} \]

4. (a) 
\[ f_X(\frac{3}{4}) = \int_{-\infty}^{\infty} f_{X,Y}(\frac{3}{4}, t)\ dt = \int_{t=0}^{\infty} \frac{3}{24} e^{-3t/4}\ dt = \frac{1}{2}. \]

(b) 
\[ f_X(5) = \int_{-\infty}^{\infty} f_{X,Y}(5, t)\ dt = 0. \]

(c) 
\[ f_X(t) = \begin{cases} 
\frac{1}{2} & \text{if } 0 < t < 2 \\
0 & \text{else} 
\end{cases} \]