1. **15 points** This is a question about the Markov property. Define sets

\[ A \overset{\text{def}}{=} \{ X_3 = k_3 \}, \quad B \overset{\text{def}}{=} \{ X_2 = k_2 \}, \quad \text{and} \quad C \overset{\text{def}}{=} \{ X_1 = k_1 \}. \]

We have two ways of understanding the Markov property. Define

a) \( \mathbb{P}(A | B \cap C) = \mathbb{P}(A | B) \)

b) \( \mathbb{P}(A \cap C | B) = \mathbb{P}(A | B) \mathbb{P}(C | B) \).

The direct definition involves the requirement a). We also have a conditional independence claim; i.e., b). We here show that these two ideas are equivalent. Namely, for any sets \( A, B, \) and \( C \) on any probability space (removing ourselves from the setup of Markov processes), show that a) and b) are equivalent.

2. **30 points** Consider the Gambler's ruin problem with \( p_{n,n+1} = p \) for \( n \geq 0 \) and \( p_{n,n-1} = q = 1 - p \) for \( n \geq 1 \). Let \( H = H^{(0)} \). Assume that \( q > p \) so that \( \mathbb{P}_n \{ H < \infty \} = 1 \).

(a) **15 points** Compute \( \mathbb{E}_n[H] \) for all \( n \geq 0 \).

(b) **15 points** Using the Laplace transform for the law of \( H \) under \( \mathbb{P}_1 \), compute \( \mathbb{E}_1[H^2] \).

3. **15 points** Define \( \tau \overset{\text{def}}{=} \inf \{ n \geq 1 : X_n = X_0 \} \). This is the *first return time*. Suppose that \( X \) is a Markov chain with transition matrix

\[
P = \begin{pmatrix}
\frac{1}{6} & \frac{2}{6} & \frac{3}{6} \\
\frac{3}{6} & \frac{1}{6} & \frac{2}{6} \\
\frac{2}{6} & \frac{2}{6} & \frac{2}{6}
\end{pmatrix}.
\]

Compute \( \mathbb{E}_1[\tau] \).

4. **15 points** Let \( \tau \) be a stopping time and let \( A \) be a fixed subset of the state space. Define \( \tau' \overset{\text{def}}{=} \inf \{ n \geq \tau : X_n \in A \} \). Show that \( \tau' \) is a stopping time.