Math 561, Section F1, Spring 2012
Final, Part II, May 8

Show all work; partial credit will be given
No calculators allowed
Do all computations up to the point where you would reach for a calculator
Remember that I have to grade this; be neat
The exam has 1 pages of questions
100 points total

1. 25 points Let ξ be a standard uniform random variable; i.e.,

$$\mathbb{P}\{\xi \in A\} = \mathfrak{L}(A \cap [0,1])$$

for all $A \in \mathscr{B}(A)$, where $\mathfrak{L}$ is standard Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

(a) 10 points Compute

$$M(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[e^{\theta\xi}\right] \qquad \theta \in \mathbb{R}$$

(b) 15 points Compute

$$I(x) \stackrel{\text{def}}{=} \sup_{\theta\in\mathbb{R}} \{\theta x - \ln M(\theta)\}$$

2. 25 points Suppose that $\lim_{n\to\infty} \mu_n = \mu$ for some $\{\mu_n\}_{n\in\mathbb{N}}$ and μ in $\mathscr{P}(\mathbb{R})$. Suppose also that

$$\sup_{n\in\mathbb{N}} \int_{x\in\mathbb{R}} |x|^{1.04}\mu_n(dx) \qquad \int_{x\in\mathbb{R}} |x|^{1.04}\mu(dx)$$

are finite. Show that

$$\lim_{n\to\infty} \int_{x\in\mathbb{R}} x\mu_n(dx) = \int_{x\in\mathbb{R}} x\mu(dx).$$

3. 25 points Suppose that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of random variables such that

$$\mathbb{P}\left\{|X_n| \ge \frac{1}{n}\right\} \le \frac{1}{2^n}.$$

Show that $\lim_{n\to\infty} X_n = 0$ $\mathbb{P}$-a.s.

4. 25 points Define $\Omega \stackrel{\text{def}}{=} [0,1)$, and $\mathscr{F} \stackrel{\text{def}}{=} \mathscr{B}[0,1)$. Let $\mathbb{P}(A) \stackrel{\text{def}}{=} \mathfrak{L}(A)$ for all $A \in \mathscr{F}$. Fix a bounded Borel-measurable function $f : [0,1] \to \mathbb{R}$ and define $X(\omega) \stackrel{\text{def}}{=} f(\omega)$ for all $\omega \in \Omega$. Define

$$\mathscr{F}_N \stackrel{\text{def}}{=} \sigma\{[k2^{-n}, (k+1)2^{-n}) : n \le N,\ k2^{-n} \le \tfrac{1}{2}\}$$
$$X_N \stackrel{\text{def}}{=} \mathbb{E}[X|\mathscr{F}_N]$$

for all $N \in \mathbb{N}$.

(a) 15 points Compute X_N for all $N \in \mathbb{N}$

(b) 10 points Compute $X_\infty \stackrel{\text{def}}{=} \lim_{N\to\infty} X_N$.