Math 461 Test 1, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 4 red, 6 blue and 10 green balls. 3 balls are randomly selected from the urn, find the probability that they are all of the same color if (a) the balls are drawn without replacement; (b) the balls are drawn with replacement.

Solution (a) If the balls are drawn without replacement, then

$$\begin{aligned} P(\text{ all 3 are of the same color}) &= \frac{\binom{4}{3}+\binom{6}{3}+\binom{10}{3}}{\binom{20}{3}} \\ &= \frac{4\cdot 3\cdot 2+6\cdot 5\cdot 4+10\cdot 9\cdot 8}{20\cdot 19\cdot 18}. \end{aligned}$$

(b) If the balls are drawn with replacement, then

$$P(\text{ all 3 are of the same color}) = \frac{4^3+6^3+10^3}{20^3}.$$

2. (20 points) An 8-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains the ace and king of at least one suit.

Solution Let A_1 be the event that the hand contains the ace and king of hearts, A_2 the event that the hand contains the ace and king of spades, A_3 the event that the hand contains the ace and king of clubs and A_4 the event that the hand contains the ace and king of diamonds. Then, by using the inclusion-exclusion formula, the probability that the hand contains the ace and king of at least one suit is equal to

$$P(\cup_{i=1}^4 A_i) = 4\frac{\binom{50}{6}}{\binom{52}{8}} - \binom{4}{2}\frac{\binom{48}{4}}{\binom{52}{8}} + \binom{4}{3}\frac{\binom{46}{2}}{\binom{52}{8}} + \frac{1}{\binom{52}{8}}.$$

3. (20 points) A box contains 7 red and 13 blue balls. Two balls are randomly selected (without replacement) and are discarded without their colors being seen. A third ball is drawn randomly. (a) Find the probability that the third ball is red. (b) Given that the third ball is red, find the probability that both discarded balls were blue.

Solution (a)

$$\begin{aligned} P(R_3) \quad &= P(R_1R_2R_3) + P(R_1B_2R_3) + P(B_1R_2R_3) + P(B_1B_2R_3) \\ &= \frac{7}{20}\frac{6}{19}\frac{5}{18} + \frac{7}{20}\frac{13}{19}\frac{6}{18} + \frac{13}{20}\frac{7}{19}\frac{6}{18} + \frac{13}{20}\frac{12}{19}\frac{7}{18} = \frac{2}{20}. \end{aligned}$$

(b)

$$P(B_1B_2|R_3) = \frac{P(B_1B_2R_3)}{P(R_3)} = \frac{\frac{13}{20}\frac{12}{19}\frac{7}{18}}{\frac{2}{20}}.$$

4. (15 points) A parallel system functions whenever at least one of its components works. Consider a parallel system with 3 components and suppose that each component independently works with probability $\frac{2}{3}$. Given that the system is functioning, find the probability the first two components are both functioning.

Solution For $i = 1, 2, 3$, let A_i be the event that the i-th component is functioning and let A be the event that the system is functioning. Then $A = A_1 \cup A_2 \cup A_3$. Thus

$$P(A) = P(A_1 \cup A_2 \cup A_3) = 1 - P(A_1^c A_2^c A_3^c) = 1 - (\frac{1}{3})^3.$$

$$P(A_1A_2|A) = \frac{P(A_1A_2A)}{P(A)} = \frac{P(A_1A_2)}{P(A)} = \frac{(\frac{2}{3})^2}{1 - (\frac{1}{3})^3}.$$

5. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ 1/2 & 1 \le x < 2, \\ \frac{x}{12} + \frac{1}{2}, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(X < 2)$; (b) $P(X = 2)$; (c) $P(1 \le X < 3)$; (d) $P(X > 3/2)$; (e) $P(2 < X \le 7)$.

Solution (a) $P(X < 2) = F(2-) = \frac{1}{2}$.
(b) $P(X = 2) = F(2) - F(2-) = (\frac{2}{12} + \frac{1}{2}) - \frac{1}{2} = \frac{1}{6}$.
(c) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{3}{12} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(d) $P(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.
(e) $P(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{2}{12} + \frac{1}{2}) = \frac{1}{3}$.

6. (15 points) A card is drawn at random from an an ordinary deck of 52 cards and its face value is noted, and then this card is returned to the deck. This procedure is done 4 times all together. Let X be the total number of aces selected and $Y = \cos(\frac{\pi}{2}X)$
(a) Find $P(X \ge 1)$.
(b) Find the expectation and variance of Y.

Solution (a) $P(X \ge 1) = 1 - P(X = 0) = 1 - (\frac{12}{13})^4$.
(b)

$$EY = E[\cos(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \cos(\frac{k\pi}{2}) \begin{pmatrix} 4 \\ k \end{pmatrix} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = (\frac{12}{13})^4 - \begin{pmatrix} 4 \\ 2 \end{pmatrix} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4.$$

$$E[Y^2] = E[\cos^2(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \cos^2(\frac{k\pi}{2}) \begin{pmatrix} 4 \\ k \end{pmatrix} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = (\frac{12}{13})^4 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4.$$

$$\mathrm{Var}(Y) = E[Y^2] - (EY)^2 = (\frac{12}{13})^4 + \begin{pmatrix} 4 \\ 2 \end{pmatrix} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4 - \left((\frac{12}{13})^4 - \begin{pmatrix} 4 \\ 2 \end{pmatrix} (\frac{1}{13})^2 (\frac{12}{13})^2 + (\frac{1}{13})^4 \right)^2.$$

Math 461 Test 2, Spring 2005

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1. (10 points) The time (in hours) required to repair a machine is an exponential random variable with parameter $\lambda = \frac{1}{2}$. Find (a) the probability that the repair time exceeds 2 hours; (b) the conditional probability that the repair time is at least 10 hours, given that its duration exceeds 9 hours.

2. (10 points) A certain basketball player knows that on average he will make 80 percent of his free throw attempts. Use normal approximation to find the probability that in 100 attempts he will be successful at least 90 times.

3. (10 points) Teams A and B play a series of games; the series will end when one of the teams wins 4 games. Suppose that team A wins each game with probability $\frac{2}{3}$, independent of the outcomes of the other games. Find the probability that a total of 6 games are played.

4. (15 points) Let X and Y be independent random variables each uniformly distributed on $(0, 1)$. Find density function of $Z = |X - Y|$.

5. (15 points) Suppose X and Y are independent geometric random variables with parameter $p = \frac{1}{3}$. Find $P(X \leq Y)$.

6. (10 points) Suppose that X is uniformly distributed over the interval $(-1, 1)$. Find the density of $Y = -\ln(1 - |X|)$.

7. (20 points) The joint density of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} . \end{cases}$$

(a) Find the marginal density of X. (b) Find EX and $\text{Var}(X)$.

8. (10 points) The gross daily sales at a certain convenient store is a normal random variable with mean \$2000 and standard deviation \$200. Assume that sales are independent from day to day. Find the probability that the total gross sales in the next 4 days exceeds \$8800.

1. (a) $P(X > 2) = \int_2^\infty \frac{1}{2}e^{-\frac{x}{2}}dx = e^{-1}$..
(b) Using the memoryless property of exponential random variables, we get

$$P(X > 10|X > 9) = P(X > 1) = \int_1^\infty \frac{1}{2}e^{-\frac{x}{2}}dx = e^{-\frac{1}{2}}.$$

2. Let X be the number of times that he will be successful in 100 attempts. Then X is a binomial random variable with parameter $n = 100$ and $p = .8$. Thus, by using normal approximation, we get

$$P(X \geq 90) = P(X \geq 89.5) = P(\frac{X - 80}{4} \geq \frac{9.5}{4}) \approx 1 - \Phi(2.37) = .0089.$$

3. Let A be the event that team A wins in 6 games, B be the event that B wins in 6 games, and C be the event that a total of 6 games are played. Then

$$P(C) = P(A) + P(B) = \binom{5}{3}(\frac{2}{3})^4(\frac{1}{3})^2 + \binom{5}{3}(\frac{1}{3})^4(\frac{2}{3})^2.$$

4. Z is a random variable taking values in $(0, 1)$. For any $z \in (0, 1)$, by geometrical considerations, we can easily get

$$P(Z \leq z) = P(|X - Y| \leq z) = 1 - (1 - z)^2.$$

Thus the density of Z is

$$f_Z(z) == \begin{cases} 2(1 - z), & z \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

5.

$$P(X \leq Y) = \sum_1^\infty P(X = k, X \leq Y) = \sum_{k=1}^\infty P(X = k, Y \geq k) = \sum_{k=1}^\infty P(X = k)P(Y \geq k)$$
$$= \sum_{k=1}^\infty (\frac{2}{3})^{k-1}\frac{1}{3}(\frac{2}{3})^{k-1} = \frac{1}{3}\sum_{k=1}^\infty (\frac{4}{9})^{k-1} = \frac{1}{3}\sum_{j=0}^\infty (\frac{4}{9})^j = \frac{1}{3} \cdot \frac{9}{5} = \frac{3}{5}.$$

6. Y is a positive random variable. For any $y > 0$,

$$P(Y \leq y) = P(-\ln(1 - |X|) \leq y) = P(1 - |X| \geq e^{-y}) = P(|X| \leq 1 - e^{-y}) = 1 - e^{-y}.$$

Thus the density of Y is given

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

7. (a). For any $x \in (0, 1)$,

$$f_X(x) = \int_0^1 (x + y)dy = x + \frac{1}{2}.$$

Thus the marginal density of X is

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

(b) $EX = \int_0^1 x(x+\frac{1}{2})dx = \frac{7}{12}$, $E(X^2) = \int_0^1 x^2(x+\frac{1}{2})dx = \frac{5}{12}$. Thus

$$\text{Var}(X) = E(X^2) - (EX)^2 = \frac{11}{144}.$$

8. From the assumption of the problem we know that the total gross sales X in the next 4 days is a normal random variable with $\mu = 8000$ and $\sigma^2 = 4 \cdot (200)^2$. Thus

$$P(X \geq 8800) = P(\frac{X - 8000}{400} \geq 2) = 1 - \Phi(2) = .0228.$$

Math 461 Final, Spring 2005

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Show all work to qualify for full credits

1. (21 points) (a) Suppose that A_1, A_2, A_3 and A_4 are independent events with $P(A_1) = 1/2$, $P(A_2) = 1/3$, $P(A_3) = 1/4$ and $P(A_4) = 1/5$. Find $P((A_1 \cap A_2) \cup (A_3 \cap A_4))$.

(b) Suppose that X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find $E[(X - 2Y)^2]$.

(c) Suppose that X, Y and Z are independent random variables, X is Poisson with parameter $\lambda_1 = 1$, Y is geometric with parameter $p = 1/3$, and Z is exponential with parameter $\lambda_2 = 2$. Find $\mathrm{Cov}(X - 2Y + Z, 2X - 3Y - 2Z)$.

2. (12 points) Two cards are randomly selected, without replacement, from an ordinary deck of 52 cards. Find the probability that one of the cards is an ace and the other card is either a 10, a jack, a queen or a king.

3. (13 points) 5 balls are randomly chosen, without replacement, from a box containing 7 red, 7 white and 7 blue balls. Find the probability that at least one of the 3 colors is missing from the chosen balls.

4. (12 points) Box A contains 2 white and 4 red balls, whereas box B contains 1 white and 1 red ball. A ball is randomly selected from box A and put into box B, and a ball is then randomly selected from box B. Given that a white ball is selected from box B, what is the probability that the transfered ball was white?

5. (15 points) A box contains 12 balls, of which 4 are white and 8 are black. Three players, A, B and C, successively draw from the box without replacement, A first, then B and then C, then A and so on. The winner is the first one to draw a white ball. Find the probability that A is the winner.

6. (11 points) Suppose that X is uniformly distributed in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Find the density of the random variable $Y = \tan X$.

7. (12 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ (1/2) + (x-1)/4, & 1 \le x < 2, \\ 11/12, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(1/2 \le X < 2)$; (b) $P(1 \le X \le 5/2)$; (c) $P(1 < X < 2)$.

8. (14 points) Suppose that X_1 and X_2 are independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Find the probability $P(X_1 = 40 | X_1 + X_2 = 100)$.

9. (15 points) The joint density of X and Y is

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} . \end{cases}$$

Find (a) $E[X]$; (b) $\text{Var}(X)$; (c) $\text{Cov}(X,Y)$.

10. (14 points) Suppose that X and Y are independent random variables and each is uniformly distributed in $(0,1)$. Find the probability $P(|X-Y| > \frac{1}{2})$.

11. (14 points) Let X and Y be independent random variables each geometrically distributed with parameter p. Put $Z = \min(X,Y)$. For any positive integer n, find (a) $P(Z \leq n)$; (b) $P(Z = n)$.

12. (14 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{otherwise} . \end{cases}$$

For $y \in (0,1)$, (a) find $f_{X|Y}(x|y)$; (b) $E(X|Y=y)$.

13. (15 points) According to a certain survey, 10 percent of 9-th grade boys and 20 percent of 9-th grade girls never eat breakfast. Assume that the breakfast habits of all the 9-th graders are independent. Suppose that random samples of 400 9th grade boys and 400 9-th grade girls are chosen. Use the central limit theorem (normal approximation) to find the probability that at least 135 of the 800 9-th graders never eat breakfast.

14 (18 points) 8 married couples are randomly seated at a round table. Let X be the number of married couples that are seated together. Find the expectation and variance of X. (Hint: Write X as a sum of some simpler random variables.)

1. (a)

$$\begin{aligned} P((A_1\cap A_2)\cup(A_3\cap A_4)) &= P(A_1\cap A_2)+P(A_3\cap A_4)-P(A_1\cap A_2\cap A_3\cap A_4) \\ &= \frac{1}{2}\frac{1}{3}+\frac{1}{4}\frac{1}{5}-\frac{1}{2}\frac{1}{3}\frac{1}{4}\frac{1}{5}=\frac{5}{24} \end{aligned}$$

(b)

$$\begin{aligned} E[(X-2Y)^2] &= E[X^2-4XY+4Y^2]=E[X^2]-4EXEY+4E[Y^2] \\ &= 5(\mathrm{Var}(X)+(EX)^2)-4(EX)^2=5\sigma^2+\mu^2. \end{aligned}$$

(c)

$$\begin{aligned} \mathrm{Cov}(X-2Y+Z,2X-3Y-2Z) &= 2\mathrm{Cov}(X,X)+6\mathrm{Cov}(Y,Y)-2\mathrm{Cov}(Z,Z) \\ &= 2\mathrm{Var}(X)+6\mathrm{Var}(Y)-2\mathrm{Var}(Z) \\ &= 2+36-\frac{1}{2}=37.5. \end{aligned}$$

2. The answer is

$$\frac{4\cdot 16}{\binom{52}{2}}=\frac{2\cdot 4\cdot 16}{52\cdot 51}.$$

3. Let A_1 be the event that “red is missing from the chosen balls”, A_2 is the event “white is missing from the chosen balls” and A_3 is the event that “blue is missing from the chosen balls”. Then

$$\begin{aligned} &P(A_1\cup A_2\cup A_3) \\ &= P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_1\cap A_3)-P(A_2\cap A_3)-P(A_1\cap A_2\cap A_3) \\ &= P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_1\cap A_3)-P(A_2\cap A_3) \\ &= 3\frac{\binom{14}{5}}{\binom{21}{5}}-3\frac{\binom{7}{5}}{\binom{21}{5}}. \end{aligned}$$

4. Let E be the event that “a white ball is selected from Box B” and F be the event that “the transfered ball is white”. Then

$$\begin{aligned} P(F|E) &= \frac{P(EF)}{P(E)}=\frac{P(EF)}{P(EF)+P(EF^c)} \\ &= \frac{P(F)P(E|F)}{P(F)P(E|F)+P(F^c)P(E|F^c)} \\ &= \frac{\frac{2}{6}\frac{2}{3}}{\frac{2}{6}\frac{2}{3}+\frac{4}{6}\frac{1}{3}}=\frac{1}{2}. \end{aligned}$$

5.For any i let B_i be the event that the i-th picked ball is black and W_i be the event that that the i-th picked ball is white. Then the probability that A is the winner is equl to

$$P(W_1)+P(B_1B_2B_3W_4)+P(B_1B_2B_3B_4B_5B_6W_7)=\frac{4}{12}+\frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9}+\frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6}.$$

6. For any real number y, we have

$$P(Y\le y)=P(\tan X\le y)=P(X\le\arctan y)=\frac{1}{2}+\frac{1}{\pi}\arctan y.$$

Thus the density of Y is given by

$$f_Y(y)=\frac{1}{\pi}\frac{1}{1+y^2},\qquad -\infty<y<\infty.$$

7. (a) $P(\frac{1}{2}\le X<2)=F(2-)-F(\frac{1}{2}-)=\frac{1}{2}+\frac{1}{4}-\frac{1}{8}=\frac{5}{8}$.

(b) $P(1\le X\le\frac{5}{2})=F(\frac{5}{2})-F(1-)=\frac{11}{12}-\frac{1}{4}=\frac{2}{3}$.

(c) $P(1<X<2)=F(2-1)-F(1)=\frac{1}{2}+\frac{1}{4}-\frac{1}{2}=\frac{1}{4}$.

8.

$$\begin{aligned}P(X_1=40|X_1+X_2=100) &= \frac{P(X_1=40,X_1+X_2=100)}{P(X_1+X_2=100)}=\frac{P(X_1=40,X_2=60)}{P(X_1+X_2=100)}\\ &= \frac{P(X_1=40)P(X_2=60)}{P(X_1+X_2=100)}=\frac{e^{-1}\frac{1}{40!}e^{-2}\frac{2^{60}}{60!}}{e^{-3}\frac{3^{100}}{100!}}\\ &= \binom{100}{40}\left(\frac{1}{3}\right)^{40}\left(\frac{2}{3}\right)^{60}.\end{aligned}$$

9. (a)

$$EX=\int_0^1\int_0^1 x(x+y)dxdy=\int_0^1(\frac{1}{3}+\frac{y}{2})dy=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}.$$

(b)

$$E[X^2]=\int_0^1\int_0^1 x^2(x+y)dxdy\int_0^1(\frac{1}{4}+\frac{y}{3})dy=\frac{1}{4}+\frac{1}{6}=\frac{5}{12},$$

therefore $\mathrm{Var}(X)=\frac{5}{12}-(\frac{7}{12})^2=\frac{11}{144}$.

(c)

$$E[XY]=\int_0^1\int_0^1 xy(x+y)dxdy=\int_0^1(\frac{y}{3}+\frac{y^2}{2})dy=\frac{1}{6}+\frac{1}{6}=\frac{1}{3},$$

therefore $\mathrm{Cov}(X,Y)=\frac{1}{3}-(\frac{7}{12})^2-\frac{1}{144}$.

10. By geometric considerations we can easily get that

$$P(|X-Y|>\frac{1}{2})=\frac{1}{4}.$$

11. (a) For any positive integer n,

$$P(\min(X,Y) > z) = P(X > n, Y > n) = P(X > n)P(Y > n) = (1-p)^{2n},$$

and so

$$P(\min(X,Y) \le n) = 1 - (1-p)^{2n}.$$

Therefore $\min(X,Y)$ is a geometric random variable with parameter $1-(1-p)^2 = 2p - p^2$.

(b) For any positive integer n, $P(\min(X,Y) = n) = (1-p)^{2(n-1)}(2p-p^2)$.

12. (a) For $y \in (0,1)$,

$$f_Y(y) = \int_0^y 2dx = 2y,$$

and so

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $E(X|Y=y) = \frac{y}{2}$.

13. Let X be the number of boys among the 400 9-th grade boys who never eats breakfast, Y be the number of girls among the 400 9-th grade girls who never eats breakfast. Then X and Y are independent. By the central limit theorem, we know that X is approximately normal with mean 40 and variance 36, and that Y is approximately normal with mean 80 and variance 64. Thus $X+Y$ is approximately normal with mean 120 and variance 100. Therefore

$$P(X+Y \ge 135) = P(X+Y \ge 134.5) = P(\frac{X+Y-120}{10} \ge \frac{14.5}{10}) = 1 - \Phi(1.45) = .0735.$$

14. For $i = 1, \ldots, 8$, let X_i equal to 1 if the i-th couple are seated together and $X_i = 0$ otherwise. Then $\sum_{i=1}^8 X_i$ is the number of couples that are seated together. For any $i = 1, \ldots, 8$, we have $P(X-i=1) = 2/15$, thus

$$EX_i = \frac{2}{15}, \quad \text{Var}(X_i) = \frac{2}{15}(1-\frac{2}{15}).$$

For $i \ne j$, we have

$$P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1) = \frac{2}{15}\frac{2}{14}.$$

Thus we have

$$\text{Cov}(X_i, X_j) = \frac{2}{15}\frac{2}{14} - \left(\frac{2}{15}\right)^2.$$

Consequently

$$E(\sum_{i=1}^8 X_i) = 8\frac{2}{15} = \frac{16}{15}$$

and

$$\text{Var}(\sum_{i=1}^8 X_i) = 8\frac{2}{15}(1-\frac{2}{15}) + 56\left(\frac{2}{15}\frac{2}{14} - \left(\frac{2}{15}\right)^2\right).$$