Math 461 Test 1, Spring 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) An urn contains 5 red, 6 blue and 8 green balls. 3 balls are randomly selected from the urn, find the probability that they are all of the same color if (a) the balls are drawn without replacement; (b) the balls are drawn with replacement.

Solution (a) If the balls are drawn without replacement, then

$$\begin{aligned} P(\text{ all 3 are of the same color}) &= \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}} \\ &= \frac{5\cdot 4\cdot 3+6\cdot 5\cdot 4+8\cdot 7\cdot 6}{19\cdot 18\cdot 17}. \end{aligned}$$

(b) If the balls are drawn with replacement, then

$$P(\text{ all 3 are of the same color}) = \frac{5^3+6^3+8^3}{19^3}.$$

2. (15 points) An 7-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains the ace and king of at least one suit.

Solution Let A_1 be the event that the hand contains the ace and king of hearts, A_2 the event that the hand contains the ace and king of spades, A_3 the event that the hand contains the ace and king of clubs and A_4 the event that the hand contains the ace and king of diamonds. Then, by using the inclusion-exclusion formula, the probability that the hand contains the ace and king of at least one suit is equal to

$$P(\cup_{i=1}^4 A_i) = 4\frac{\binom{50}{5}}{\binom{52}{7}} - \binom{4}{2}\frac{\binom{48}{3}}{\binom{52}{7}} + \binom{4}{3}\frac{\binom{46}{1}}{\binom{52}{7}}.$$

3. (20 points) A box contains 7 red and 13 blue balls. Two balls are randomly selected (without replacement) and are discarded without their colors being seen. A third ball is drawn randomly. (a) Find the probability that the third ball is red. (b) Given that the third ball is red, find the probability that both discarded balls were blue.

Solution (a)

$$\begin{aligned} P(R_3) \quad &= P(R_1R_2R_3) + P(R_1B_2R_3) + P(B_1R_2R_3) + P(B_1B_2R_3) \\ &= \frac{7}{20}\frac{6}{19}\frac{5}{18} + \frac{7}{20}\frac{13}{19}\frac{6}{18} + \frac{13}{20}\frac{7}{19}\frac{6}{18} + \frac{13}{20}\frac{12}{19}\frac{7}{18} = \frac{2}{20}. \end{aligned}$$

(b)

$$P(B_1B_2|R_3) = \frac{P(B_1B_2R_3)}{P(R_3)} = \frac{\frac{13}{20}\frac{12}{19}\frac{7}{18}}{\frac{2}{20}}.$$

4. (20 points) A circuit is given below. The probability of the i-th switch is on is $\frac{1}{i+1}$, $i = 1, 2, 3, 4$. Suppose that all switches function independently. (a) Find the probability that a current can flow from A to B. (b) Given that a current can flow from A to B, find the probability that both switches 1 and 2 are on.

Solution (a) Let A be the vent that a current can flow from A to B, then the probability of A is equal to

$$P((A_1 \cup A_2 \cup A_3)A_4) = P(A_1 \cup A_2 \cup A_3)P(A_4) = (1 - P(A_1^c A_2^c A_3^c))P(A_4) = (1 - \frac{1}{2}\frac{2}{3}\frac{3}{4})\frac{1}{5} = \frac{3}{20}.$$

(b)

$$P(A_1A_2|A) = \frac{P(A_1A_2A)}{P(A)} = \frac{P(A_1A_2A_4)}{P(A)} = \frac{\frac{1}{2}\frac{1}{3}\frac{1}{5}}{\frac{3}{20}} = \frac{2}{9}.$$

5. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ 1/2 & 1 \le x < 2, \\ \frac{x}{12} + \frac{1}{2}, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(X < 2)$; (b) $P(X = 2)$; (c) $P(1 \le X < 3)$; (d) $P(X > 3/2)$; (e) $P(2 < X \le 7)$.

Solution (a) $P(X < 2) = F(2-) = \frac{1}{2}$.
(b) $P(X = 2) = F(2) - F(2-) = (\frac{2}{12} + \frac{1}{2}) - \frac{1}{2} = \frac{1}{6}$.
(c) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{3}{12} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(d) $P(X > 3/2) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$.
(e) $P(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{2}{12} + \frac{1}{2}) = \frac{1}{3}$.

6. Independent trials, each results in a success with probability $\frac{2}{3}$, are performed 4 times. Let X be the total number of successes and $Y = \sin(\frac{\pi}{2}X)$
(a) (5 points) Find $P(X \ge 1)$.
(b) (10 points) Find the expectation and variance of Y.

Solution (a) $P(X \ge 1) = 1 - P(X = 0) = 1 - (\frac{1}{3})^4$.
(b)

$$EY = E[\sin(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \sin(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = \binom{4}{1} (\frac{2}{3})(\frac{1}{3})^3 - \binom{4}{1} (\frac{2}{3})^3 (\frac{1}{3}) = -\frac{8}{27}.$$

$$E[Y^2] = E[\sin^2(\frac{\pi}{2}X)] = \sum_{k=0}^{4} \sin^2(\frac{k\pi}{2}) \binom{4}{k} (\frac{2}{3})^k (\frac{1}{3})^{4-k} = \binom{4}{1} (\frac{2}{3})(\frac{1}{3})^3 + \binom{4}{1} (\frac{2}{3})^3 (\frac{1}{3}) = \frac{40}{81}.$$

$$\text{Var}(Y) = E[Y^2] - (EY)^2 = \frac{40}{81} - (\frac{8}{27})^2.$$

Math 461 Test 2, Spring 2005

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1. (10 points) A machine produces screws, 1 percent of which are defective. Use Poisson approximation to find the probability that in a box of 100 screws there is at most 1 defective.

2. (10 points) A certain basketball player knows that on average he will make 80 percent of his free throw attempts. Use normal approximation to find the probability that in 100 attempts he will be successful at least 90 times.

3. (10 points) Teams A and B play a series of games; the series will end when one of the teams wins 4 games. Suppose that team A wins each game with probability $\frac{2}{3}$, independent of the outcomes of the other games. Find the probability that a total of 6 games are played.

4. (20 points) Let X and Y be independent random variables each uniformly distributed on $(0, 1)$. Find $P(Y \geq X | Y \geq \frac{1}{2})$.

5. (10 points) The time X it takes John to solve a problem is an exponential random variable with parameter $\lambda = 1$, and the time Y it take Mike to solve the same problem is also an exponential random variable with parameter $\lambda = 2$. Suppose that X and Y are independent. Find $P(X \leq Y)$.

6. (10 points) Suppose that X is uniformly distributed over the interval $(-1, 1)$. Find the density of $Y = -\ln(1 - |X|)$.

7. (20 points) The joint density of X and Y is given by

$$f(x, y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise .} \end{cases}$$

(a) Find the marginal density of X. (b) Find EX and $\text{Var}(X)$.

8. (10 points) Suppose that X and Y are independent random variables, X is Poisson with parameter λ_1 and Y is Poisson with parameter λ_2. Find $P(X = k | X + Y = 100)$ for $k = 0, 1, \ldots, 100$.

1. Let X be the number of defective ones in the box. Then X is approximately a Poisson random variable with parameter $\lambda = 1$. Thus

$$P(X \le 1) = P(X = 0) + P(X = 1) \approx 2e^{-1}.$$

2. Let X be the number of times that he will be successful in 100 attempts. Then X is a binomial random variable with parameter $n = 100$ and $p = .8$. Thus, by using normal approximation, we get

$$P(X \ge 90) = P(X \ge 89.5) = P(\frac{X - 80}{4} \ge \frac{9.5}{4}) \approx 1 - \Phi(2.37) = .0089.$$

3. Let A be the event that team A wins in 6 games, B be the event that B wins in 6 games, and C be the event that a total of 6 games are played. Then

$$P(C) = P(A) + P(B) = \binom{5}{3} (\frac{2}{3})^4 (\frac{1}{3})^2 + (5) (\frac{1}{3})^4 (\frac{2}{3})^2.$$

4. By geometrical considerations, we can easily see that $P(Y \ge \frac{1}{2}) = \frac{1}{2}$ and $P(Y \ge X, Y \ge \frac{1}{2}) = \frac{3}{8}$. Thus

$$P(Y \ge X | Y \ge \frac{1}{2}) = \frac{P(Y \ge X, Y \ge \frac{1}{2})}{P(Y \ge \frac{1}{2})} = \frac{3}{4}.$$

5.

$$P(X \le Y) = \int_0^\infty (\int_x^\infty 2e^{-x} e^{-2y} dy) dx = \int_0^\infty e^{-x} (\int_x^\infty 2e^{-2y} dy) dx = \int_0^\infty e^{-3x} dx = \frac{1}{3}.$$

6. Y is a positive random variable. For any $y > 0$,

$$P(Y \le y) = P(-\ln(1 - |X|) \le y) = P(1 - |X| \ge e^{-y}) = P(|X| \le 1 - e^{-y}) = 1 - e^{-y}.$$

Thus the density of Y is given

$$f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & y \le 0. \end{cases}$$

7. (a). For any $x \in (0, 1)$,

$$f_X(x) = \int_0^1 (x + y) dy = x + \frac{1}{2}.$$

Thus the marginal density of X is

$$f_X(x) = \begin{cases} x + \frac{1}{2}, & x \in (0, 1) \\ 0, & \text{otherwise.} \end{cases}$$

(b) $EX = \int_0^1 x(x + \frac{1}{2}) dx = \frac{7}{12}$, $E(X^2) = \int_0^1 x^2 (x + \frac{1}{2}) dx = \frac{5}{12}$. Thus

$$\text{Var}(X) = E(X^2) - (EX)^2 = \frac{11}{144}.$$

8. Since X and Y are independent Poisson random variables with parameters λ_1 and λ_2 respectively, we know that $X+Y$ is a Poisson random variable with parameter $\lambda_1+\lambda_2$. Thus, for any $k = 0, 1, \ldots, 100$,

$$P(X=k|X+Y=100) = \frac{P(X=k, X+Y=100)}{P(X+Y=100)} = \frac{P(X=k, Y=100-k)}{P(X+Y=100)}$$

$$= \frac{P(X=k(P(Y=100-k)}{P(X+Y=100)} = \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{100-k}}{(100-k)!}}{e^{-\lambda_1-\lambda_2}\frac{(\lambda_1+\lambda_2)^{100}}{100!}}$$

$$= \left(\begin{array}{c}100\\k\end{array}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{100-k}.$$

Math 461 Final, Spring 2005

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Show all work to qualify for full credits

1. (21 points) (a) Suppose that A_1, A_2, A_3 and A_4 are independent events with $P(A_1) = 1/2$, $P(A_2) = 1/3$, $P(A_3) = 1/4$ and $P(A_4) = 1/5$. Find $P(A_1 \cap (A_2 \cup A_3 \cup A_4))$.

(b) Suppose that X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find $E[(X - Y)^2]$.

(c) Suppose that X, Y and Z are independent random variables, X is Poisson with parameter $\lambda_1 = 1$, Y is geometric with parameter $p = 1/3$, and Z is exponential with parameter $\lambda_2 = 2$. Find $\mathrm{Cov}(X - 2Y + Z, 2X + 3Y - 2Z)$.

2. (12 points) Two cards are randomly selected, without replacement, from an ordinary deck of 52 cards. Find the probability that one of the cards is an ace and the other card is either a 10, a jack, a queen or a king.

3. (13 points) 5 balls are randomly chosen, without replacement, from a box containing 6 red, 6 white and 6 blue balls. Find the probability that at least one of the 3 colors is missing from the chosen balls.

4. (12 points) Box A contains 2 white and 4 red balls, whereas box B contains 1 white and 1 red ball. A ball is randomly selected from box A and put into box B, and a ball is then randomly selected from box B. Given that a white ball is selected from box B, what is the probability that the transfered ball was white?

5. (15 points) A box contains 12 balls, of which 4 are white and 8 are black. Three players, A, B and C, successively draw from the box without replacement, A first, then B and then C, then A and so on. The winner is the first one to draw a white ball. Find the probability that A is the winner.

6. (11 points) Suppose that X is uniformly distributed in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Find the density of the random variable $Y = \tan X$.

7. (12 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ (1/2) + (x-1)/4, & 1 \le x < 2, \\ 11/12, & 2 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

Find (a) $P(1/2 \le X < 2)$; (b) $P(1 \le X \le 5/2)$; (c) $P(1 < X < 2)$.

8. (14 points) Suppose that X_1 and X_2 are independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Find the probability $P(X_1 = 20 | X_1 + X_2 = 50)$.

9. (15 points) The joint density of X and Y is

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise} . \end{cases}$$

Find (a) $E[X]$; (b) $\text{Var}(X)$; (c) $\text{Cov}(X,Y)$.

10. (14 points) Suppose that X and Y are independent random variables and each is uniformly distributed in $(0,1)$. Find the probability $P(2X > Y)$.

11. (14 points) Let X and Y be independent random variables each geometrically distributed with parameter p. Put $Z = \min(X,Y)$. For any positive integer n, find (a) $P(Z \leq n)$; (b) $P(Z = n)$.

12. (14 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2, & 0 < x < y < 1, \\ 0, & \text{otherwise} . \end{cases}$$

For $y \in (0,1)$, (a) find $f_{X|Y}(x|y)$; (b) $E(X|Y = y)$.

13. (15 points) According to a certain survey, 10 percent of 9-th grade boys and 20 percent of 9-th grade girls never eat breakfast. Assume that the breakfast habits of all the 9-th graders are independent. Suppose that random samples of 400 9-th grade boys and 400 9-th grade girls are chosen. Use the central limit theorem (normal approximation) to find the probability that at least 140 of the 800 9-th graders never eat breakfast.

14 (18 points) 10 married couples are randomly seated at a round table. Let X be the number of married couples that are seated together. Find the expectation and variance of X. (Hint: Write X as a sum of some simpler random variables.)

1. (a)

$$
\begin{aligned}
P(A_1 \cap (A_2 \cup A_3 \cup A_4)) &= P(A_1)P(A_2 \cup A_3 \cup A_4) = \frac{1}{2}(1 - P(A_2^c \cap A_3^c \cap A_4^c)) \\
&= \frac{1}{2}(1 - P(A_2^c)P(A_3^c)P(A_4^c)) = \frac{1}{2}\left(1 - \frac{2}{3}\frac{3}{4}\frac{4}{5}\right) = \frac{3}{10}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[(X-Y)^2] &= E[X^2 - 2XY + Y^2] = E[X^2] - 2EXEY + E[Y^2] \\
&= 2[\mathrm{Var}(X) + (EX)^2] - 2(EX)^2 = 2(\sigma^2 + \mu^2) - 2\mu^2 = 2\sigma^2.
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathrm{Cov}(X - 2Y + Z, 2X + 3Y - 2Z) &= 2\mathrm{Cov}(X,X) - 6\mathrm{Cov}(Y,Y) - 2\mathrm{Cov}(Z,Z) \\
&= 2\mathrm{Var}(X) - 6\mathrm{Var}(Y) - 2\mathrm{Var}(Z) \\
&= 2 - 36 - \frac{1}{2} = -34.5.
\end{aligned}
$$

2. The answer is

$$
\frac{4 \cdot 16}{\binom{52}{2}} = \frac{2 \cdot 4 \cdot 16}{52 \cdot 51}.
$$

3. Let A_1 be the event that "red is missing from the chosen balls", A_2 is the event "white is missing from the chosen balls" and A_3 is the event that "blue is missing from the chosen balls". Then

$$
\begin{aligned}
&P(A_1 \cup A_2 \cup A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&= 3\frac{\binom{12}{5}}{\binom{18}{5}} - 3\frac{\binom{6}{5}}{\binom{18}{5}}.
\end{aligned}
$$

4. Let E be the event that "a white ball is selected from Box B" and F be the event that "the transfered ball is white". Then

$$
\begin{aligned}
P(F|E) &= \frac{P(EF)}{P(E)} = \frac{P(EF)}{P(EF) + P(EF^c)} \\
&= \frac{P(F)P(E|F)}{P(F)P(E|F) + P(F^c)P(E|F^c)} \\
&= \frac{\frac{2}{6}\frac{2}{3}}{\frac{2}{6}\frac{2}{3} + \frac{4}{6}\frac{1}{3}} = \frac{1}{2}.
\end{aligned}
$$

5. For any i let B_i be the event that the i-th picked ball is black and W_i be the event that that the i-th picked ball is white. Then the probability that A is the winner is equl to

$$P(W_1) + P(B_1B_2B_3W_4) + P(B_1B_2B_3B_4B_5B_6W_7) = \frac{4}{12} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{4}{9} + \frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{4}{6}.$$

6. For any real number y, we have

$$P(Y \le y) = P(\tan X \le y) = P(X \le \arctan y) = \frac{1}{2} + \frac{1}{\pi}\arctan y.$$

Thus the density of Y is given by

$$f_Y(y) = \frac{1}{\pi}\frac{1}{1+y^2}, \qquad -\infty < y < \infty.$$

7. (a) $P(\frac{1}{2} \le X < 2) = F(2-) - F(\frac{1}{2}-) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$.

(b) $P(1 \le X \le \frac{5}{2}) = F(\frac{5}{2}) - F(1-) = \frac{11}{12} - \frac{1}{4} = \frac{2}{3}$.

(c) $P(1 < X < 2) = F(2-1) - F(1) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$.

8.

$$\begin{aligned} P(X_1 = 20 | X_1 + X_2 = 50) &= \frac{P(X_1 = 20, X_1 + X_2 = 50)}{P(X_1 + X_2 = 50)} = \frac{P(X_1 = 20, X_2 = 30)}{P(X_1 + X_2 = 50)} \\ &= \frac{P(X_1 = 20)P(X_2 = 30)}{P(X_1 + X_2 = 50)} = \frac{e^{-1}\frac{1}{20!}e^{-2}\frac{2^{30}}{30!}}{e^{-3}\frac{3^{50}}{50!}} \\ &= \binom{50}{20}\left(\frac{1}{3}\right)^{20}\left(\frac{2}{3}\right)^{30}. \end{aligned}$$

9. (a)

$$EX = \int_0^1\int_0^1 x(x+y)dxdy = \int_0^1 (\frac{1}{3} + \frac{y}{2})dy = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

(b)

$$E[X^2] = \int_0^1\int_0^1 x^2(x+y)dxdy \int_0^1 (\frac{1}{4} + \frac{y}{3})dy = \frac{1}{4} + \frac{1}{6} = \frac{5}{12},$$

therefore $\mathrm{Var}(X) = \frac{5}{12} - (\frac{7}{12})^2 = \frac{11}{144}$.

(c)

$$E[XY] = \int_0^1\int_0^1 xy(x+y)dxdy = \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy = \frac{1}{6} + \frac{1}{6} = \frac{1}{3},$$

therefore $\mathrm{Cov}(X,Y) = \frac{1}{3} - (\frac{7}{12})^2 - \frac{1}{144}$.

10. By geometric considerations we can easily get that

$$P(2X > Y) = \frac{3}{4}.$$

11. (a) For any positive integer n,

$$P(\min(X,Y) > z) = P(X > n, Y > n) = P(X > n)P(Y > n) = (1-p)^{2n},$$

and so

$$P(\min(X,Y) \le n) = 1 - (1-p)^{2n}.$$

Therefore $\min(X,Y)$ is a geometric random variable with parameter $1-(1-p)^2 = 2p-p^2$.

(b) For any positive integer n, $P(\min(X,Y) = n) = (1-p)^{2(n-1)}(2p-p^2)$.

12. (a) For $y \in (0,1)$,

$$f_Y(y) = \int_0^y 2dx = 2y,$$

and so

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) $E(X|Y=y) = \frac{y}{2}$.

13. Let X be the number of boys among the 400 9-th grade boys who never eats breakfast, Y be the number of girls among the 400 9-th grade girls who never eats breakfast. Then X and Y are independent. By the central limit theorem, we know that X is approximately normal with mean 40 and variance 36, and that Y is approximately normal with mean 80 and variance 64. Thus $X+Y$ is approximately normal with mean 120 and variance 100. Therefore

$$P(X+Y \ge 140) = P(X+Y \ge 139.5) = P(\frac{X+Y-120}{10} \ge \frac{19.5}{10}) = 1 - \Phi(1.95) = .0256.$$

14. For $i = 1, \ldots, 10$, let X_i equal to 1 if the i-th couple are seated together and $X_i = 0$ otherwise. Then $\sum_{i=1}^{10} X_i$ is the number of couples that are seated together. For any $i = 1, \ldots, 10$, we have $P(X-i=1) = 2/19$, thus

$$EX_i = \frac{2}{19}, \quad \text{Var}(X_i) = \frac{2}{19}(1-\frac{2}{19}).$$

For $i \neq j$, we have

$$P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1|X_i = 1) = \frac{2}{19}\frac{2}{18}.$$

Thus we have

$$\text{Cov}(X_i, X_j) = \frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2.$$

Consequently

$$E(\sum_{i=1}^{10} X_i) = 10\frac{2}{19} = \frac{20}{19}$$

and

$$\text{Var}(\sum_{i=1}^{10} X_i) = 10\frac{2}{19}(1-\frac{2}{19}) + 90\left(\frac{2}{19}\frac{2}{18} - \left(\frac{2}{19}\right)^2\right).$$