Math 461 Test 1, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (20 points) (a) Suppose that A and B are two events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find $P(A^c \cup B)$.
(b) Suppose that E, F, G are independent event with $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{3}$ and $P(G) = \frac{1}{4}$. Find $P((E \cup F) \cap G^c)$.

(a) $P(A^c \cup B) = 1 - P(A \cap B^c) = 1 - [P(A) - P(A \cap B)] = 1 - [\frac{1}{2} - \frac{1}{4}] = \frac{3}{4}$.

(b) $P((E \cup F) \cap G^c) = P(E \cup F)P(G^c) = [P(E) + P(F) - P(E \cap F)]P(G^c) = [\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3}]\frac{3}{4} = \frac{1}{2}$.

2. (15 points) 10 people, including A, B and C, are randomly arranged in a line. Find the probability that A, B and C are together in the line.

The probability that A, B and C are together in the line is

$$\frac{8!3!}{10!} = \frac{1}{15}.$$

3. (15 points) An 9-card hand is drawn without replacement from an ordinary deck of 52 cards. Find the probability that it contains all 4 cards of at least 1 of the 13 denominations.

For $i = 1, 2, \ldots, 13$, let A_i be the event that the hand contains all 4 cards of denomination i. Then we are looking for the probability of $\cup_{i=1}^{13} A_i$. Using the inclusion-exclusion formula, we get

$$\begin{aligned} P(\cup_{i=1}^{13} A_i) &= \sum_{i=1}^{13} P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\ &= 13\frac{\binom{48}{5}}{\binom{52}{9}} - \binom{13}{3}\frac{\binom{44}{1}}{\binom{52}{9}}. \end{aligned}$$

4. (20 points) Box A has 5 white and 7 black balls. Box B has 3 white and 9 black balls. We flip a fair coin. If the coin comes up heads, we randomly select a ball from Box A, whereas if the coin comes up tails, we randomly select a ball from Box B. (a) Find the probability the selected ball is white. (b) Suppose that the selected ball is white, what is the probability that the coin came up tails?

Let H be teh event that the coin comes heads, T be the event that the coin comes up tails and W be the event that selected ball is white.
(a) $P(W) = P(W|H)\frac{1}{2} + P(W|T)\frac{1}{2} = \frac{5}{12}\frac{1}{2} + \frac{3}{12}\frac{1}{2} = \frac{1}{3}$.
(b) $P(T|W) = \frac{P(WT)}{P(W)} = \frac{P(W|T)P(T)}{P(W)} = \frac{3/24}{8/24} = \frac{3}{8}$.

5. (15 points) A box contains 4 red balls and 6 black balls. Players A and B draw balls from the box consecutively until a red ball is selected. Find the probability that B gets the red ball. (A draws first ball, then B and so on. There is no replacement of the balls drawn.)

The only cases for B to get the red ball are BR, $BBBR$ and $BBBBBR$. Therefore the answer is

$$\frac{6}{10}\frac{4}{9} + \frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{4}{7} + \frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}\frac{2}{6}\frac{4}{5}.$$

6. (15 points) We draw cards, one at a time, at random and successively from an ordinary deck of 52 cards with replacement. Find the probability that an ace appears before a face card (that is, a Jack, a Queen or a King)?

Let A be the event the first drawn card is an ace, B the event that the first drawn card is a face card, and C teh event that the first drawn card is neither an ace nor a face card. Let E be the event that an ace appears before a face card. Then, conditioning on the first draw, we get

$$P(E) = P(E|A)\frac{1}{13} + P(E|B)\frac{3}{13} + P(E|C)\frac{9}{13} = \frac{1}{13} + P(E)\frac{9}{13}.$$

Thus $P(E) = \frac{1}{4}$.

Math 461 Test 2, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (9 points) The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{1}{12}x + \frac{1}{2} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Find (a) $P(1 \le X < 3)$, (b) $P(X > \frac{3}{2})$, (c) $P(2 < X \le 7)$.

(a) $P(1 \le X < 3) = F(3-) - F(1-) = (\frac{1}{4} + \frac{1}{2}) - \frac{1}{4} = \frac{1}{2}$.
(b) $P(X > \frac{3}{2}) = 1 - F(\frac{3}{2}) = \frac{1}{2}$.
(c) $(2 < X \le 7) = F(7) - F(2) = 1 - (\frac{1}{6} + \frac{1}{2}) = \frac{1}{3}$.

2. (11 points) Suppose that X is a uniform random variable on $(-1, 1)$. Find the density of the random variable $Y = X^4$.

Y takes values in $(0, 1)$. For any $y \in (0, 1)$,

$$P(Y \le y) = P(X^4 \le y) = P(-y^{1/4} \le X \le y^{1/4}) = y^{1/4}.$$

Thus the density of Y is given by

$$g(y) = \begin{cases} \frac{1}{4}y^{-3/4} & y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

3. (20 points) A certain flight has 90 seats for passengers. Assume that any passenger has an 80% probability of showing up for that flight (with different passengers mutually independent). Suppose that 100 tickets are sold. Use the normal approximation to find the probability that the flight can accommodate all passengers (with a ticket) that show up.

Let X be the number of ticketed passengers who show up for the flight, then X is a binomial random variable with parameters $(100, .8)$. Thus by normal approximation we get that

$$P(X \le 90) = P(X < 90.5) = P(\frac{X - 80}{4} \le \frac{90.5 - 80}{4}) = \Phi(2.62) = .9955.$$

4. (20 points) Two teams play a series of games and they stop playing as soon as one of the teams wins 4 games. Suppose that the two teams are evenly matched and each has probability $\frac{1}{2}$ of winning each game. Find the expected number of games played.

$$P(X=4) = 2 \cdot (\frac{1}{2})^4 = \frac{1}{8}$$
$$P(X=5) = 2 \cdot \binom{4}{1} (\frac{1}{2})^5 = \frac{1}{4}$$
$$P(X=6) = 2 \cdot \binom{5}{2} (\frac{1}{2})^6 = \frac{5}{16}$$
$$P(X=7) = 2 \cdot \binom{6}{3} (\frac{1}{2})^7 = \frac{5}{16}$$

Thus
$$E[X] = 4\frac{1}{8} + 5\frac{1}{4} + 6\frac{5}{16} + 7\frac{5}{16} = \frac{93}{16}.$$

5. (20 points) Let X be an exponential random variable with parameter $\lambda = \ln 3$. Define a positive integer valued random variable Y letting $Y = n$ whenever $n-1 < X \leq n$ for any positive integer n. (a) Find the mass function of Y; (b) find $E[Y]$ and $\text{Var}(Y)$.

(a) For any positive integer n,

$$P(Y=n) = P(n-1 < X \leq n) = e^{-(n-1)\ln 3} - e^{-n \ln 3} = (\frac{1}{3})^{n-1} - (\frac{1}{3})^n = (\frac{1}{3})^{n-1}\frac{2}{3}.$$

(b) From part (a) we know that Y is a geometric random variable with parameter $p = \frac{2}{3}$, thus $E[y] = \frac{3}{2}$ and $\text{Var}(Y) = \frac{3}{4}$.

6. (20 points) The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} ye^{-2x} & x > 0, 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density of X; (b) find $E[X]$ and $\text{Var}(X)$.

(a) The marginal density of X is

$$f_X(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) From part (a) we know that X is an exponential random variable with parameter $\lambda = 2$. Thus $E[X] = \frac{1}{2}$ and $\text{Var}(X) = \frac{1}{4}$.

Math 461 Final, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (9 points) (a) Suppose that A_1, A_2, A_3 and A_4 are independent events with $P(A_1) = 1/2$, $P(A_2) = 1/3$, $P(A_3) = 1/4$ and $P(A_4) = 1/5$. Find $P((A_1 \cup A_2) \cap (A_3 \cup A_4))$.

$$\begin{aligned}
&P((A_1 \cup A_2) \cap (A_3 \cup A_4)) = P(A_1 \cup A_2)P(A_3 \cup A_4) \\
&= (P(A_1) + P(A_2) - P(A_1 \cap A_2))(P(A_3) + P(A_4) - P(A_3 \cap A_4)) \\
&= (\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3})(\frac{1}{4} + \frac{1}{5} - \frac{1}{4} \cdot \frac{1}{5}) \\
&= \frac{2}{3}\frac{2}{5} = \frac{4}{15}.
\end{aligned}$$

(b) Suppose that X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find $E[(X - 2Y)^2]$.

$$\begin{aligned}
&E[(X - 2Y)^2] = E[X^2] - 4E[XY] + 4E[Y^2] = 5E[X^2] - 4[EX]^2 \\
&= 5(E[X^2] - [EX]^2) + [EX]^2 = 5\sigma^2 + \mu^2.
\end{aligned}$$

(c) Suppose that X, Y and Z are independent random variables, X is Poisson with parameter $\lambda_1 = 1$, Y is geometric with parameter $p = 1/3$, and Z is exponential with parameter $\lambda_2 = 2$. Find $\mathrm{Cov}(X - 2Y + Z, 2X - 3Y - 2Z)$.

$$\begin{aligned}
&\mathrm{Cov}(X - 2Y + Z, 2X - 3Y - 2Z) = \mathrm{Cov}(X, 2X) + \mathrm{Cov}(-2Y, -3Y) + \mathrm{Cov}(Z, -2Z) \\
&= 2\mathrm{Var}(X) + 6\mathrm{Var}(Y) - 2\mathrm{Var}(Z) \\
&= 2 \cdot 1 + 6 \cdot 6 - 2\frac{1}{4} = 37.5.
\end{aligned}$$

2. (6 points) A closet has 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be exactly one complete pair?

$$\frac{10 \cdot \binom{9}{6} \cdot 2^6}{\binom{20}{8}}.$$

3. (7 points) If 5 married couples are lined up at random in a straight line, find the probability that no wife is next to her husband.

For $i = 1, \dots, 5$, let E_i be the even that the i-th couple are together. Then the probability we are looking for is $1 - P(\cup_{i=1}^5 E_i)$. By the inclusion-exclusion formula we know that

$$\begin{aligned} P(\cup_{i=1}^5 E_i) &= \sum_{i=1}^{5} P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1}E_{i_2}E_{i_3}) \\ &- \sum_{i_1<i_2<i_3<i_4} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}) + P(\cap_{i=1}^5 E_i) \\ &= 5 \cdot \frac{2 \cdot 9!}{(10)!} - \binom{5}{2} \cdot \frac{2^2 \cdot 8!}{(10)!} + \binom{5}{3} \cdot \frac{2^3 \cdot 7!}{(10)!} - \binom{5}{4} \cdot \frac{2^4 \cdot 6!}{(10)!} + \frac{2^5 \cdot 5!}{(10)!}. \end{aligned}$$

So the desired anwser is

$$1 - 5 \cdot \frac{2 \cdot 9!}{(10)!} + \binom{5}{2} \cdot \frac{2^2 \cdot 8!}{(10)!} - \binom{5}{3} \cdot \frac{2^3 \cdot 7!}{(10)!} + \binom{5}{4} \cdot \frac{2^4 \cdot 6!}{(10)!} - \frac{2^5 \cdot 5!}{(10)!}.$$

4. (7 points) A box initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the box along with 2 other balls of the same color. Find the probability that of the first three balls, exactly 2 are black.

With the obvious notation, the probability we are looking for is equal to $P(W_1B_2B_3) + P(B_1W_2B_3) + P(B_1B_2W_3)$.

$$\begin{aligned} P(W_1B_2B_3) &= \frac{5}{12}\frac{7}{14}\frac{9}{16} \\ P(B_1W_2B_3) &= \frac{7}{12}\frac{5}{14}\frac{9}{16} \\ P(B_1B_2W_3) &= \frac{7}{12}\frac{9}{14}\frac{5}{16} \end{aligned}$$

So the desired probability is

$$\frac{5 \cdot 7 \cdot 9}{4 \cdot 14 \cdot 16}.$$

5. (7 points) Independent trials, each results in a success with probability p and failure with probability $1 - p$, are performed. Let X be the total number of failures before the 5th success. Find the mass function of X.

X is a nonnegative integer-valued random variable. For any $x = 0, 1 \dots,$

$$P(X = x) = \binom{4+x}{4} p^5 (1-p)^x.$$

6. (6 points) Suppose that X is uniformly distributed in $(-2,2)$. Find the density of the random variable $Y = X^2$.

Y takes values in $[0,4)$. For any $y \in (0,4)$,

$$P(Y \leq y) = P(X^2 \leq y) = P(-y^{1/2} \leq X \leq y^{1/2}) = \frac{y^{1/2}}{2}.$$

Thus the density of y is given by

$$g(y) = \begin{cases} \frac{1}{4}y^{-1/2}, & y \in (0,4) \\ 0, & \text{otherwise} \end{cases}$$

7. (8 points) Suppose that X_1 and X_2 are independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Find the probability $P(X_1 = 40|X_1 + X_2 = 100)$.

$X_1 + X_2$ is a Poisson random variable with parameter 3. So

$$\begin{aligned} P(X_1 = 40|X_1 + X_2 = 100) &= \frac{P(X_1 = 40, X_1 + X_2 = 100)}{P(X_1 + X_2 = 100)} \\ &= \frac{P(X_1 = 40, X_2 = 60)}{P(X_1 + X_2 = 100)} \\ &= \frac{P(X_1 = 40)P(X_2 = 60)}{P(X_1 + X_2 = 100)} \\ &= \binom{100}{40} (\frac{1}{3})^{40} (\frac{2}{3})^{60}. \end{aligned}$$

8. (8 points) Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X,Y)$.

$$\begin{aligned} E[X] &= \int_0^1 \int_0^1 x\frac{3}{2}(x^2 + y^2)dxdy = \frac{5}{8} \\ E[Y] &= \int_0^1 \int_0^1 y\frac{3}{2}(x^2 + y^2)dxdy = \frac{5}{8} \\ E[XY] &= \int_0^1 \int_0^1 xy\frac{3}{2}(x^2 + y^2)dxdy = \frac{3}{8} \end{aligned}$$

Therefore

$$\text{Cov}(X,Y) = \frac{3}{8} - (\frac{5}{8})^2.$$

9. (8 points) Suppose that X and Y are independent random variables. If X is an exponential random variable with parameter $\lambda_1 = 1$ and Y is an exponential random variable with parameter $\lambda_2 = 2$, find the probability $P(X \geq Y)$.

$$\begin{aligned} P(X \geq Y) \quad &= \int_0^\infty \left(\int_y^\infty e^{-x} 2e^{-2y} dx \right) dy \\ &= \int_0^\infty 2e^{-3y} dy = \frac{2}{3}. \end{aligned}$$

10. (8 points) Let X and Y be independent random variables. Suppose that X is a geometric random variable with parameter $p_1 = \frac{1}{2}$ and Y is a geometric random variable with parameter $p_2 = \frac{1}{3}$. Find the probability $P(X = Y)$.

$$\begin{aligned} P(X = Y) \quad &= \sum_{i=1}^\infty P(X = i, X = Y) = \sum_{i=1}^\infty P(X = i, Y = i) \\ &= \sum_{i=1}^\infty P(X = i)P(Y = i) = \sum_{i=1}^\infty \frac{1}{2}(\frac{1}{2})^{i-1} \frac{1}{3}(\frac{2}{3})^{i-1} \\ &= \frac{1}{6} \sum_{j=0}^\infty (\frac{1}{3})^j = \frac{1}{6}\frac{3}{2} = \frac{1}{4}. \end{aligned}$$

11. (8 points) Suppose that U and V are independent random variables and both are uniformly distributed in $(0, 1)$. Define $X = \min(U, V)$ and $Y = \max(U, V)$. For $y \in (0, 1)$, find $E(X|Y = y)$.

The joint density of X and Y is given by

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

So the marginal density of Y is

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and consequently for $y \in (0, 1)$,

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 < x < y \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $E(X|Y = y) = \frac{y}{2}$.

12. (8 points) Both player A and B try 100 free throws. It is known that, on average, A will make 80 percent of his free throw attempts and B will make 90 percent of his free throw attempts. Use the central limit theorem to find the probability at least 175 of these 200 free throw attempts will be successful.

By the central limit theorem, the number of free throws X made by A is approximately normal with mean 80 and variance 16, and the number of free throws Y made by A is approximately normal with mean 90 and variance 9. So the total number of free throws $X+Y$ is approximately normal with mean 170 and variance 25. Thus

$$P(X+Y \geq 170) = P(X+Y \geq 174.5) = P(\frac{X+Y-170}{5} \geq \frac{4.5}{5}) = 1 - \Phi(.9) = .1841.$$

13. (10 points) For a group of 100 people, find the expected number of days of the year that are birthdays of exactly 4 people.

For $= 1, \cdots, 365$, let $X_i = 1$ if day i is the birthdays of exactly 4 people and $X_i = 0$ otherwise. We are looking for $E(\sum_{i=1}^{365} X_i$ Since

$$P(X_i = 1) = \frac{\binom{100}{4}(364)^{96}}{(365)^{100}},$$

The anwser we are looking for is

$$365 \cdot \frac{\binom{100}{4}(364)^{96}}{(365)^{100}}.$$