Math 361 Test 1, Fall 2003

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. Suppose that A, B and C are independent events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{5}$.
(a) (7 points) Find $P(A \cap B \cap C)$.
(b) (7 points) Find $P(A \cap (B \cup C))$.
2. An 8-card hand is drawn without replacement from an ordinary deck.
(a) (8 points) Find the probability that the hand contains at least one hearts.
(b) (8 points) Find the probability that the hand contains two 3-of-a-kinds and no pairs or 4-of a kinds. (This means that the face values of the cards are of the form x, x, x, y, y, y, u, v, where x, y, u, v are distinct face values)
3. (10 points) Cards are randomly chosen from an ordinary deck one by one without replacement until a king appears. Find the probability that this occurs with the 5th chosen card.
4. (10 points) An urn contains 3 red and 5 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A gets the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)
5. A box initially contains 10 black balls and 10 white balls. Each time a ball is selected, its color noted and it is returned to the box along with 2 additional balls of the same color.
(a) (8 points) Find the probability that the second and third balls are white.
(b) (7 points) Find the conditional probability that the first ball is black given that the second and third balls are white.
6. In an ordinary deck of 52 cards there are 4 kings. A card is drawn at random from the deck and its face value noted; then this card is returned to the deck. This procedure is done 5 times all together.
(a) (8 points) Find the probability that there are at least two kings in those selected cards;
(b) (7 points) Find the probability that there are exactly two kings in the selected cards given that there are at least two kings in those selected cards.
7. An urn contains 10 black balls, 10 white balls, 10 red balls and 10 blue balls. 10 balls are chosen randomly from the urn without replacement. Find the probability that
(a) (5 points) there are no black balls chosen;
(b) (5 points) there are no black balls and no white balls chosen;
(c) (5 points) there are no black, no white and no red balls chosen;
(d) (5 points) there is at least one ball of each of the 4 colors among the chosen balls.

1. (a) $P(A \cap B \cap B \cap C) = P(A)P(B)P(C) = \frac{1}{60}$.
(b) $P(A \cap (B \cup C)) = P(A)P(B \cup C) = \frac{1}{3}(P(B) + P(C) - P(B \cap C)) = \frac{1}{3}(\frac{1}{4} + \frac{1}{5} - \frac{1}{4}\frac{1}{5}) = \frac{2}{15}$.

2. (a) Let A be the event that the hand contains at least one hearts. Then

$$P(A) = 1 - P(A^c) = \frac{\binom{39}{8}}{\binom{52}{8}}.$$

(b) Let B be the event that the hand contains 2 3-of-a-kinds and no pairs or 4-of-a-kinds. Then

$$P(B) = \frac{\binom{13}{2}\binom{4}{3}^2\binom{11}{2}\binom{4}{1}^2}{\binom{52}{8}}.$$

3. The answer is

$$\frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}.$$

4. Let A be the event that "A gets the red ball". Then

$$P(A) = P(R_1) + P(B_1B_2R_3) + P(B_1B_2B_3B_4R_5) = \frac{3}{8} + \frac{5}{8}\frac{4}{7}\frac{3}{6} + \frac{5}{8}\frac{4}{7}\frac{3}{6}\frac{2}{5}\frac{3}{4}.$$

5. (a) $P(R_2R_3) = P(R_1R_2R_3) + P(B_1R_2R_3) = \frac{10}{20}\frac{12}{22}\frac{14}{24} + \frac{10}{20}\frac{10}{22}\frac{12}{24}$.
(b) $P(B_1|R_2R_3) = \frac{P(B_1R_2R_3)}{P(R_2R_3)} = \frac{\frac{10}{20}\frac{10}{22}\frac{12}{24}}{\frac{10}{20}\frac{12}{22}\frac{14}{24} + \frac{10}{20}\frac{10}{22}\frac{12}{24}}$.

6. Let A be the event that "there are at least two kings in the selected cards" and let B be the event that "there are exactly two kings in the selected cards".
(a) $P(A) = 1 - (\frac{48}{52})^5 - 5 \cdot (\frac{48}{52})^4 \frac{4}{52}$.
(b) $P(B|A) = \frac{P(B)}{P(A)} = \frac{\binom{5}{2}(\frac{48}{52})^3(\frac{4}{52})^2}{1 - (\frac{48}{52})^5 - 5 \cdot (\frac{48}{52})^4 \frac{4}{52}}$.

7. Let A_1 be the event that "there are no black balls chosen", A_2 be the event that "there are no white balls chosen", A_3 be the event that "there are no red balls chosen" and A_4 be the event that "there are no blue balls chosen". Then
(a) $P(A_1) = \frac{\binom{30}{10}}{\binom{40}{10}}$.
(b) $P(A_1A_2) = \frac{\binom{20}{10}}{\binom{40}{10}}$.

(c) $P(A_1A_2A_3) = \dfrac{\binom{10}{10}}{\binom{40}{10}}$.

(d) The complement of the event $A_1 \cup A_2 \cup A_3 \cup A_4$ is the event that "there is at least one ball of each color". So the answer is

$$
\begin{aligned}
& 1 - P(A_1 \cup A_2 \cup A_3 \cup A_4) \\
& = 1 - \left(4 \cdot \frac{\binom{30}{10}}{\binom{40}{10}} - \binom{4}{2} \frac{\binom{20}{10}}{\binom{40}{10}} + \binom{4}{3} \frac{\binom{10}{10}}{\binom{40}{10}} \right).
\end{aligned}
$$

Math 361 Test 2, Fall 2003

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/3, & 0 \le x < 1, \\ x/2 & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

Find (a) $P(1/2 \le X \le 3/2)$; (b) $P(1/2 \le X \le 1)$; (c) $P(1/2 \le X < 1)$; (d) $P(1 \le X \le 3/2)$; (e) $P(1 < X < 2)$.

2. (13 points) Suppose that a biased coin that lands on heads with probability $\frac{2}{3}$ is flipped 15 times. Given that a total of 10 heads result, find the conditional probability that the first 4 outcomes are H, T, T, H (meaning that the first flip is heads, the second is tails, the third is tails, and the fourth is heads).

3. (13 points) A certain typing agency has two typists. The average number of errors per article is 3 when typed by the first typist and 4 when typed by the second. If your article is equally likely to be typed by either typist, approximate the probability that it will have at most 1 error.

4. (14 points) An urn 6 white and 6 black balls. We randomly select 4 balls without replacement. If two of them are white and two of them are black, we stop. If not, we return the balls to the urn and again random select 4 balls. This continues until exactly 2 of the 4 chosen are white. Let X be the number of trials needed. (a) Find the mass function of X; (b) Find the expectation of X.

5. (10 points) Suppose that X is uniformly distributed over the interval $(-1, 1)$. Find the density of $Y = X^4$.

6. (13 points) Buses arrive at a specified stop at 15 minutes intervals starting at 7 am. That is, they arrive at 7, 7:15, 7:30, and so on. Suppose that a passenger arrives at the stop X minutes after 7 am, where X is uniformly distributed in $(0, 30)$. Let Y be the number of minutes that the passenger has to wait for a bus. Find the expectation of Y.

7. (14 points) Suppose that X is an exponential random variable with parameter $\lambda = 2$. Find (a) $\mathrm{Var}(1 - 2X)$; (b) $E[(2 + X)^2]$.

8. (13 points) A certain airplane has 85 seats for passengers. Assume that any particular passenger has an 80% probability of showing up for a certain flight (with different passengers mutually independent in the appropriate sense). Suppose that 100 tickets are sold for the flight. Find the approximate probability that the flight is not over-sold (that is, all the passengers who show up can be accommodated).

1. (a) $P(\frac{1}{2} \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}-) = \frac{3}{4} - \frac{1}{6}$.
(b) $P(\frac{1}{2} \le X \le 1) = F(1) - F(\frac{1}{2}-) = \frac{1}{2} - \frac{1}{6}$.
(c) $P(\frac{1}{2} \le X < 1) = F(1-) - F(\frac{1}{2}-) = \frac{1}{3} - \frac{1}{6}$.
(d) $P(1 \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(1-) = \frac{3}{4} - \frac{1}{3}$.
(e) $P(1 < X < 2) = F(2-) - F(1) = 1 - \frac{1}{2}$.

2. Let A be the event that "there are exactly 10 heads in the 15 trials", B be the event that "the first 4 outcomes are H, T, T, H", and C be the event that "there are exactly 8 heads in trials number 5 through 15". Then

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)P(C)}{P(A)} = \frac{(\frac{2}{3})^2(\frac{1}{3})^2\binom{11}{8}(\frac{2}{3})^8(\frac{1}{3})^3}{\binom{15}{10}(\frac{2}{3})^{10}(\frac{1}{3})^5} = \frac{\binom{11}{8}}{\binom{15}{10}}.$$

3. Let A be the event that "your article has at most 1 error", B be the event that "your article is typed by the first typist" and C the event that "your article is typed by the second typist". Then

$$P(A) = \frac{1}{2}P(A|B) + \frac{1}{2}P(A|C) = \frac{1}{2}(e^{-3} + 3e^{-3}) + \frac{1}{2}(e^{-4} + 4e^{-4}).$$

4. Let $p = \binom{6}{2}\binom{6}{2}/\binom{12}{4}$. Then
(a) for $x = 1, 2, \ldots$,

$$P(X = x) = (1-p)^{x-1}p.$$

X is a geometric random variable with parameter p.
(b) $EX = \frac{1}{p}$.

5. For $y \le 0$, $P(Y \le y) = 0$. For $y \ge 1$, $P(Y \le y) = 1$. For $y \in (0,1)$,

$$P(Y \le y) = P(X^4 \le y) = P(-y^{1/4} \le X \le y^{1/4}) = y^{1/4}.$$

So the density of $Y = X^4$ is

$$f(y) = \begin{cases} \frac{1}{4}y^{-3/4}, & y \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

6. The random variable Y can be expressed in terms of X as follows:

$$Y = \begin{cases} 15 - X, & 0 < X \le 15 \\ 30 - X, & 15 < X \le 30. \end{cases}$$

So

$$EY = \int_0^{15} (15 - x)\frac{1}{30}dx + \int_{15}^{30} (30 - x)\frac{1}{30}dx = \frac{15}{2}.$$

7. (a) $\mathrm{Var}(1-2X) = 4\mathrm{Var}(X) = 1$.

(b) $E[(2+X)^2] = E[4+4X+X^2] = 4+4Ex+E[X^2] = 4+2+(\mathrm{Var}(X)+(EX)^2) = 6+(\frac{1}{4}+\frac{1}{4}) = 6.5$.

8. Let S_{100} be the total number of passengers showing up for the flight. Then

$$\begin{aligned} P(S_{100} \le 85) &= P(S_{100} \le 85.5) = P(\frac{S_{100}-80}{\sqrt{100\cdot\frac{4}{5}\frac{1}{4}}} \le \frac{85.5-80}{\sqrt{100\cdot\frac{4}{5}\frac{1}{4}}}) \\ &= P(\frac{S_{100}-80}{\sqrt{100\cdot\frac{4}{5}\frac{1}{4}}} \le \frac{5.5}{4}) \approx \Phi(1.37) = .914. \end{aligned}$$

Math 361 Test 3, Fall 2003

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (14 points) Suppose that X is an exponential random variable with parameter $\lambda = 1$, Y is a geometric random variable with parameter $p = 1/2$ and that X and Y are independent. Find (a) $\mathrm{Var}(4X - 2Y)$, (b) $\mathrm{Cov}(2X - Y, X + 3Y)$.
2. (14 points) Let X and Y be independent random variables each uniformly distributed in $(-1, 1)$. Find $P(|X - Y| \le 1)$.
3. (10 points) Suppose that X, Y and Z are independent Poisson random variables with parameter $\lambda = 1$. Find $P(X + Y + Z \le 2)$.
4. (14 points) The joint density function of X and Y is

$$f(x,y) = \begin{cases} x + y, & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise .} \end{cases}$$

Find $P(X + Y < 1)$.
5. (14 points) Let $X_1, \ldots, X_{100}$ be independent exponentially distributed random variables with parameter $\lambda = 1$. Use the central limit theorem to find $P(X_1 + \ldots + X_{100} < 110)$.
6. (14 points) The joint density of X and Y is given by

$$f(x,y) = \begin{cases} 2, & 0 < x \le y < 1 \\ 0, & \text{otherwise .} \end{cases}$$

(a) Find the conditional density of Y given $X = x$ for $x \in (0, 1)$. (b) Find $E(Y|X = \frac{1}{2})$.
7. (20 points) A total of 10 balls are distributed into 10 boxes in such a way that each ball is equally likely to be put into any of the 10 boxes. Let X be the number of empty boxes. (a) Find EX; (b) $\mathrm{Var}(X)$.

1. (a) $\mathrm{Var}(4X-2Y)=\mathrm{Var}(4X)+\mathrm{Var}(-2Y)=16\mathrm{Var}(X)+4\mathrm{Var}(Y)=24$.
(b) $\mathrm{Cov}(2X-Y,X+3Y)=2\mathrm{Cov}(X,X)-\mathrm{Cov}(Y,X)-6\mathrm{Cov}(X,Y)-3\mathrm{Cov}(Y,Y)=2\mathrm{Var}(X)-3\mathrm{Var}(Y)=-4$.

2. By geometric considerations, we can easily get that $P(|X-Y|\le 1)=\frac{3}{4}$.

3. $X+Y+Z$ is Poisson random variable with parameter $\lambda=3$, so $P(X+Y+Z\le 2)=e^{-3}+3e^{-3}+\frac{9}{2}e^{-3}$.

4.

$$\begin{aligned}P(X+Y<1)=) &= \int_0^1\int_0^{1-x}(x+y)dydx=\int_0^1\left(\frac{(1-x)^2}{2}+x(1-x)\right)dx\\ &= \int_0^1(\frac{1}{2}-\frac{1}{2}x^2)dx=\frac{1}{3}.\end{aligned}$$

5. For $i=1,\dots,100$, $EX_i=1$ and $\mathrm{Var}(X_i)=1$. Thus

$$\begin{aligned}P(X_1+\cdots+X_{100}<110) &= P(\frac{X_1+\cdots+X_{100}-100}{\sqrt{100}}<\frac{110-100}{\sqrt{100}})\\ &= P(\frac{X_1+\cdots+X_{100}-100}{10}<1)\approx\Phi(1)=.8413.\end{aligned}$$

6. (a). For $x\in(0,1)$,

$$f_X(x)=\int_x^1 2dy=2(1-x),$$

so

$$f_{Y|X}(y|x)=\begin{cases}\frac{1}{1-x}, & y\in(x,1)\\ 0, & \text{otherwise}\end{cases}$$

(b) $E(Y|X=\frac{1}{2})=\int_{1/2}^1 2ydy=\frac{3}{4}$.

7. For $i=1,\dots,10$, define $X_i=1$ if box number i is empty and $X_i=0$ otherwise. Then $X=X_1+\cdots+X_{10}$. Since $P(X-i=1)=(\frac{9}{10})^{10}$, we have

$$E(X_i)=(\frac{9}{10})^{10},\quad \mathrm{Var}(X_i)=(\frac{9}{10})^{10}\left(1-(\frac{9}{10})^{10}\right).$$

For $i\neq j$, we have $P(X_i=1,X_j=1)=(\frac{8}{10})^{10}$, so

$$E(X_iX_i)=(\frac{8}{10})^{10},$$

and hence

$$\mathrm{Cov}(X_i,X_j)=(\frac{8}{10})^{10}-(\frac{9}{10})^{20}.$$

Therefore

$$\begin{aligned}\mathrm{Var}(X) &= \sum_{i=1}^{10}\mathrm{Var}(X_i)+\sum_{i\neq j}\mathrm{Cov}(X_i,X_j)\\ &= 10\cdot(\frac{9}{10})^{10}\left(1-(\frac{9}{10})^{10}\right)+90\cdot\left((\frac{8}{10})^{10}-(\frac{9}{10})^{20}\right).\end{aligned}$$

Math 361 Final, Fall 2003

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (12 points) (a) Suppose that A_1, A_2 and A_3 are independent events with $P(A_1) = 1/2$, $P(A_2) = 1/3$, $P(A_3) = 1/4$. Find $P(A_1 \cup (A_2 \cap A_3))$.
(b) Suppose that X is a random variable with $EX = 1$ and $\text{Var}(X) = 5$. Find $E[(2+X)^2]$.
(c) Suppose that X, Y and Z are independent random variables, X is Poisson with parameter $\lambda = 1$, Y is geometric with parameter $p = 1/2$, and Z is normal with parameters $\mu = 0$ and $\sigma^2 = 1$. Find $\text{Cov}(X - 2Y, 3Y - Z)$.
2. (10 points) A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be no complete pair?
3. (14 points) A hand of 13 cards is randomly chosen from an ordinary deck of 52 cards. Find the probability that the hand contains all 4 of at least one of the 13 denominations.
4. (10 points) Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a fair coin. If the outcome of the coin flip is heads, we select a ball from urn A, whereas if the outcome is tails, we select a ball from urn B. Suppose that a white ball is selected. Find the probability that coin landed tails.
5. (10 points) A certain flight uses a plane which has 40 seats and 45 tickets has been sold for this flight. Assume that each passenger will show up for this flight with probability .95, independent of other passengers. Find the probability that all passengers (who have purchased tickets for this flight) showing up for this flight can be accommodated.
6. (12 points) Let X and Y be independent random variables uniformly distributed on $(0, 1)$. Find the distribution and the density of the random variable $Z = |X - Y|$.
7. (14 points) A set of 100 people, consisting of 50 men and 50 women, is randomly divided into 50 pairs of 2 each. Let X be the number of pairs which consist of a man and a woman. Find EX and $\text{Var}(X)$. (Hint: For $i = 1, \ldots, 50$, define $X_i = 1$ if the i-man is paired with a woman and $X_i = 0$ otherwise.)
8. (12 points) Suppose that X and Y are independent geometric random variables with parameter $p = 1/2$. Find the probability $P(X + Y = 100)$.
9. (12 points) The joint density of X and Y is

$$f(x,y) = \begin{cases} \frac{x}{5} + \frac{y}{20}, & 0 < x < 1, 1 < y < 5, \\ 0, & \text{otherwise} . \end{cases}$$

Find (a) $E[X]$; (b) $\text{Var}(X)$; (c) $\text{Cov}(X, Y)$.
10. (10 points) Suppose that X and Y are independent random variables. Given that X is normal with parameters $\mu_1 = 2$ and $\sigma_1^2 = 1$, Y is normal with parameters $\mu_2 = 1$ and $\sigma_2^2 = 3$. Find the probability $P(X > Y)$.
11. (12 points) Suppose that U and V are independent exponential random variables with parameter $\lambda = 1$. (a) Find the density of $X = \min(U, V)$; (b) find $E[X]$.

12. (12 points) The joint density of X and Y is

$$f(x,y) = \begin{cases} e^{-x/y}e^{-y}y^{-1}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{otherwise .} \end{cases}$$

For $y > 0$, (a) find $f_{X|Y}(x|y)$; (b) $E(X|Y=y)$.

13. (10 points) Suppose that X_i, $i = 1, \ldots, 100$ are independent and identically distributed random variables with $E[X_i] = 2$ and $\mathrm{Var}(X_i) = 4$. Use the central limit theorem to find an approximation of the probability $P(X_1 + \ldots + X_{100} \geq 220)$.

1. (a)

$$\begin{aligned} P(A_1 \cup (A_2 \cap A_3)) &= P(A_1) + P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3) \\ &= P(A_1) + P(A_2)P(A_3) - P(A_1)P(A_2)P(A_3) = \frac{13}{24}. \end{aligned}$$

(b)

$$\begin{aligned} E((2+X)^2) &= E(4 + 4X + X^2) = 4 + 4EX + E(X^2) \\ &= 4 + 4 + (\mathrm{Var}(X) + (EX)^2) = 14. \end{aligned}$$

(C)

$$\begin{aligned} \mathrm{Cov}(X - 2Y, 3Y - Z) &= 3\mathrm{Cov}(X, Y) - 6\mathrm{Cov}(Y, Y) - \mathrm{Cov}(X, Z) + 2\mathrm{Cov}(Y, Z) \\ &= -6\mathrm{Var}(Y) = -12. \end{aligned}$$

2.

$$\frac{20 \cdot 18 \cdot 16 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 6}{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13}.$$

3. For $i = 1, \ldots, 13$, let A_i be the event that all 4 cards of denomination i are included in the hand. Here Ace is regarded as 1, Jack as 11, Queen as 12 and King as 13. Then

$$\begin{aligned} P(\cup_{i=1}^{13} A_i) &= \sum^{13} i = 1 P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \\ &= 13 \cdot \frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2} \cdot \frac{\binom{44}{5}}{\binom{52}{13}} + \binom{13}{3} \cdot \frac{\binom{40}{1}}{\binom{52}{13}} \end{aligned}$$

4. Let H be the event that "the coin turns up heads", T the event "the coin turns up tailss", and W the event that "a white ball is selected". Then $P(H) = P(T) = \frac{1}{2}$ and $P(W|H) = \frac{5}{12}$, $P(W|T) = \frac{3}{15}$. Thus

$$\begin{aligned} P(H|W) &= = \frac{P(WH)}{P(W)} = \frac{P(WH)}{P(WH) + P(WT)} \\ &= \frac{P(H)P(W|H)}{P(H)P(W|H) + P(T)P(W|T)} \\ &= \frac{\frac{5}{12} \cdot \frac{1}{2}}{\frac{5}{12} \cdot \frac{1}{2} + \frac{3}{15} \cdot \frac{1}{2}}. \end{aligned}$$

5. Let X be the number of ticketed passengers who show up for the flight. Then

$$\begin{aligned} P(X \leq 40) &= 1 - (P(X = 41) + P(X = 42) + P(X = 43) + P(X = 44) + P(X = 45)) \\ &= 1 - \left(\binom{45}{41} (.95)^41(.05)^4 + \binom{45}{42} (.95)^42(.05)^3 \right) \\ &\quad - \left(\binom{45}{43} (.95)^43(.05)^2 + \binom{45}{44} (.95)^44(.05) + (.95)^45 \right). \end{aligned}$$

6. Z is a random variable taking values in $(0,1)$. So we know that for $z \le 0$, $P(Z \le z) = 0$ and for $z \ge 1$, $P(Z \le z) = 1$. By geometric considerations we see that for $z \in (0,1)$, $P(Z \le z) = P(|X - Y| \le z) = 1 - (1-z)^2$. Thus the density of Z is

$$f_Z(z) = \begin{cases} 2(1-z), & z \in (0,1) \\ 0, & \text{otherwsie} \end{cases}$$

7. X can be written as $X = \sum_{i=1}^{50} X_i$. For $i = 1, \ldots 50$, $E(X_i) = P(X_i = 1) = \frac{50}{99}$ and $\mathrm{Var}(X_i) = \frac{50}{99}\frac{49}{99}$, so

$$EX = \sum_{i=1}^{50} E(X_i) = 50 \cdot \frac{50}{99}.$$

For $i \ne j$, $E(X_iX_j) = P(X_i = 1, X_j = 1) = \frac{50}{99} \cdot \frac{49}{97}$, and so

$$\begin{aligned} \mathrm{Var}(X) &= \sum_{i=1}^{50} \mathrm{Var}(X_i) + \sum_{i \ne j} \mathrm{Cov}(X_i, X_j) \\ &= 50 \cdot \frac{50}{99}\frac{49}{99} + 50 \cdot 49 \cdot \left(\frac{50}{99} \cdot \frac{49}{97} - (\frac{50}{99})^2 \right). \end{aligned}$$

8.

$$\begin{aligned} P(X+Y=100) &= \sum_{i=1}^{99} P(X=i, X+Y=100) = \sum_{i=1}^{99} P(X=i, Y=100-i) \\ &= \sum_{i=1}^{99} P(X=i)P(Y=100-i) = \sum_{i=1}^{99} \frac{1}{2}(\frac{1}{2})^{i-1}\frac{1}{2}(\frac{1}{2})^{99-i} \\ &= \sum_{i=1}^{99} (\frac{1}{2})^{100} = 99 \cdot (\frac{1}{2})^{100}. \end{aligned}$$

9.

$$\begin{aligned} EX &= \int_1^5 \int_0^1 x(\frac{x}{5} + \frac{y}{20})dxdy = \frac{17}{30}, \\ EY &= \int_1^5 \int_0^1 y(\frac{x}{5} + \frac{y}{20})dxdy = \frac{98}{30}, \\ E(X^2) &= \int_1^5 \int_0^1 x^2(\frac{x}{5} + \frac{y}{20})dxdy = \frac{2}{5}. \end{aligned}$$

Thus

$$\mathrm{Var}(X) = \frac{2}{5} - (\frac{17}{30})^2.$$

Since

$$E(XY) = \int_1^5 \int_0^1 xy(\frac{x}{5} + \frac{y}{20})dxdy = \frac{11}{6},$$

we get that

$$\mathrm{Cov}(X,Y) = \frac{11}{6} - \frac{17}{30} \cdot \frac{98}{30}.$$

10. We know that $X-Y$ is a normal random variable with parameters $mu=1$ and $\sigma^2=4$. Thus

$$
\begin{aligned}
P(X>Y) &= P(X-Y>0)=1-P(X-Y\leq 0)\\
&= 1-P(\frac{X-Y-1}{2}\leq -\frac{1}{2})\\
&= 1-\Phi(-\frac{1}{2})=\Phi(\frac{1}{2})=.6915.
\end{aligned}
$$

11. For any $x>0$,

$$
\begin{aligned}
P(X\leq x) &= P(\min(U,V)\leq x)=1-P(\min(U,V)>x)\\
&= 1-P(U>x,V>x)=1-P(U>x)P(V>x)\\
&= 1-e^{-2x},
\end{aligned}
$$

so

$$
f_X(x)=\begin{cases} 2e^{-2x}, & x>0\\ 0, & \text{otherwise}\end{cases}
$$

This is an exponential density with parameter $\lambda=2$, so $EX=\frac{1}{2}$.

12. For $y>0$,

$$
F_Y(y)=\int_0^\infty f(x,y)dx=\int_0^\infty \frac{1}{y}e^{-x/y}e^{-y}dy=e^{-y},
$$

so

$$
f_{X|Y}(x|y)=\begin{cases} \frac{1}{y}e^{-x/y}, & x>0\\ 0, & \text{otherwise}\end{cases}
$$

This is an exponential density with parameter $\lambda=\frac{1}{y}$, so

$$
E(X|Y=y)=y.
$$

13.

$$
\begin{aligned}
P(\sum_{i=1}^{100}\geq 220) &= 1-P(\sum_{i=1}^{100}<220)=1-P(\frac{\sum_{i=1}^{100}-200}{20}<\frac{220-200}{20})\\
&= 1-\Phi(1)=.1587.
\end{aligned}
$$