Math 461, Section D13, Fall 2010
Quiz 1, August 27 **Name:** ______________________

1. $\boxed{\text{10 points}}$ (taken from question 4 on p. 16) Allen, Bob, Christie, and Danielle are going to form a band. There are four positions: vocals, percussion, guitar, and kazoo.

   (a) $\boxed{\text{5 points}}$ Assume that all of the band members can do all positions in the band, How many different arrangements are possible?

   (b) $\boxed{\text{5 points}}$ Assume that Allen can only play percussion or kazoo. How many arrangements are possible?

Answers

1. (a) $4!$

   (b) $2 \times 3!$

Math 461, Section D13, Fall 2010
Quiz 2, September 1 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that we have 8 women, and 15 men. Two of the men are named Alex and Bob. Suppose that we form a committee of 4 women and 6 men. What is the probability that Alex and Bob do not serve together? (To make sense of this, suppose that Alex and Bob do not get along. If they are not on the committee together, the committee will be a peaceful one. We are interested in the probability that the committee will be a peaceful one).

Answers

1.

$$1 - \frac{\binom{8}{4}\binom{13}{4}}{\binom{8}{4}\binom{15}{6}} = 1 - \frac{binom134}{\binom{15}{6}}.$$

Math 461, Section D13, Fall 2010
Quiz 3, September 10 **Name:** ____________________

1. $\boxed{\text{10 points}}$ (taken from question 3.6 on p. 102) A box contains 9 white and 7 red balls. Sample 6 times without replacement. What is the probability that the first and fourth balls are white given that you picked exactly 4 white balls?

Answers

1.

$$\frac{(9)_4(7)_2\binom{4}{2}}{(9)_4(7)_2\binom{6}{4}} = \frac{\binom{4}{2}}{\binom{6}{4}}$$

Math 461, Section D13, Fall 2010
Quiz 4, September 15 **Name:** ____________________

1. $\boxed{\text{10 points}}$ (taken from question 3.35 on p. 104) It's your birthday! Not only do you get to take a 461 present, but your mother or father has hidden a present for you. With probability .6, the present was hidden by your father. If your father hides a present, it he hides it upstairs 70% of the time, while if your mother hides a present, it is equally likely to be upstairs or downstairs.
   (a) $\boxed{\text{5 points}}$ What is the probability that the present is downstairs?
   (b) $\boxed{\text{5 points}}$ If the present is upstairs, what is the probability that it was hidden by your mother?

Answers

1.

$$U = \{\text{upstairs}\}, \qquad F = \{\text{hidden by father}\}.$$

(a)

$$\mathbb{P}(U^c) = \mathbb{P}(U^c|F)\mathbb{P}(F) + \mathbb{P}(U^c|F^c)\mathbb{P}(F^c) = .3 \times .6 + .5 \times .4.$$

(b)

$$\mathbb{P}(F^c|U) = \frac{\mathbb{P}(U|F^c)\mathbb{P}(F^c)}{\mathbb{P}(U|F)\mathbb{P}(F) + \mathbb{P}(U|F^c)\mathbb{P}(F^c)} = \frac{.7 \times .6}{.7 \times .6 + .5 \times .4}.$$

Math 461, Section D13, Fall 2010

Quiz 5, September 29 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that we toss a sequence of independent coins, and that $\mathbb{P}\{\text{heads}\} = \frac{4}{5}$ for each coin. Let $X = 1$ if the first heads occurs on the 5th toss, and $X = 0$ otherwise. Compute the probability mass function p_X of X.

Answers

1.

$$p_X(j) = \begin{cases} \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right) & \text{if } j = 1 \\ 1 - \left(\frac{4}{5}\right)^4 \left(\frac{1}{5}\right) & \text{if } j = 0 \\ 0 & \text{else} \end{cases}$$

Math 461, Section D13, Fall 2010
Quiz 6, October 1 **Name:** ____________

1. $\boxed{\text{10 points}}$ Suppose that we toss a sequence of coins, where $\mathbb{P}\{\text{heads}\} = p$ for some $p \in (0,1)$. Let X be the position of the first heads. Define the function $f(z) \stackrel{\text{def}}{=} \min\{z, 10\}$, and define $Y \stackrel{\text{def}}{=} \min\{X, 10\}$.
   (a) $\boxed{\text{3 points}}$ Graph f.
   (b) $\boxed{\text{7 points}}$ Compute the probability mass function of Y.

Answers

(a) $f(z)=z$ for $z\le 10$, and $f(z)=10$ for $z\ge 10$.

(b)

$$p_Y(j)=\begin{cases}(1-p)^{j-1}p & \text{if } j\in\{1,2\dots 9\}\\ (1-p)^9 & \text{if } j=10\\ 0 & \text{else}\end{cases}$$

Math 461, Section D13, Fall 2010
Quiz 7, October 13 Name: ____________________

1. $\boxed{\text{10 points}}$ Let X be a random variable such that

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[\exp\left[\theta X\right]\right] = \exp\left[\frac{1}{2}\theta^2 + 5\theta\right]$$

for all $\theta \in \mathbb{R}$.

   (a) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X]$.

   (b) $\boxed{\text{5 points}}$ Compute the variance of X.

For $\theta < 5$, we have that

$$\varphi_X'(\theta) = (\theta + 5)\varphi_X(\theta) \qquad \text{and} \qquad \varphi_X''(\theta) = \left\{1 + (\theta + 5)^2\right\}\varphi_X(\theta).$$

(a) $\mathbb{E}[X] = \varphi_X'(0) = 5$.

(b) $\mathrm{V}(X) = \varphi_X''(\theta) - (\varphi_X'(\theta))^2 = 1 + 5^2 - 5^2 = 1$.

Math 461, Section D13, Fall 2010
Quiz 8, October 25 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Let X be a random variable with density

$$f_X(t) = \begin{cases} \frac{1}{4} & \text{if } 0 < t < 1 \\ \frac{3}{8} & \text{if } 1 < t < 3 \\ 0 & \text{else} \end{cases}$$

Compute the cumulative distribution function $F_X(t) \overset{\text{def}}{=} \mathbb{P}\{X \le t\}$ of X

If $t<0$, $F_X(t)=0$, and if $t\ge 3$, then $F_X(t)=1$. If $0\le t<1$, then

$$F_X(t)=\int_{s=0}^t \frac{1}{4}ds=\frac{1}{4}t.$$

If $1\le t<3$, then

$$F_X(t)=\int_{s=0}^1 \frac{1}{4}ds+\int_{s=1}^t \frac{3}{8}ds=\frac{1}{4}+\frac{3}{8}(t-1).$$

Thus

$$F_X(t)=\begin{cases}0 & \text{if } t<0\\ \frac{1}{4}t & \text{if } 0\le t<1\\ \frac{1}{4}+\frac{3}{8}(t-1) & \text{if } 1\le t<3\\ 1 & \text{if } t\ge 3\end{cases}$$

Math 461, Section D13, Fall 2010
Quiz 9, November 12 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Let X be a Gaussian random variable with mean 5 and variance 36. Find the value of c such that $\mathbb{P}\{X \geq c\} = .4$.

We can write $X = 6Z + 5$, where Z is a standard normal. Thus we want

$$.4 = \mathbb{P}\{X \geq c\} = \mathbb{P}\{6Z+5 \geq c\} = \mathbb{P}\left\{Z \geq \frac{c-5}{6}\right\} = 1-\mathbb{P}\left\{Z \geq \frac{c-5}{6}\right\} = 1-\Phi\left(\frac{c-5}{6}\right).$$

Hence we want that

$$\Phi\left(\frac{c-5}{6}\right) = .6;$$

so (approximately)

$$\frac{c-5}{6} = .26$$

so

$$c = 6 \times .26 + 5 = 6.56.$$

Math 461, Section D13, Fall 2010
Quiz 10, November 15 **Name:** ______________________

1. $\boxed{\text{10 points}}$ Fix $p \in (0,1)$. Suppose that X and Y have joint density $p_{X,Y}$ where for all integers i and j

$$p_{X,Y}(i,j) = \begin{cases} (1-p)^{i+j-2}p^2 & \text{if } 1 \le i < j \\ (1-p)^{2i-2}p & \text{if } 1 \le i = j \end{cases}$$

Compute $p_X(6)$.

Answers

1. We have that

$$
\begin{aligned}
p_X(6) &= \sum_j p_{X,Y}(6,j) = p_{X,Y}(6,6) + \sum_{j=7}^\infty p_{X,Y}(6,j) = (1-p)^{2\times 6-2}p + \sum_{j=7}^\infty (1-p)^{4+j}p^2 \\
&= (1-p)^{10}p + (1-p)^4p^2\sum_{j=7}^\infty (1-p)^j = (1-p)^{10}p + (1-p)^4p^2\frac{(1-p)^7}{p} \\
&= (1-p)^{10}p + (1-p)^{11}p = (1-p)^{10}p\left\{2-p\right\}.
\end{aligned}
$$

Math 461, Section D13, Fall 2010
Quiz 11, December 3 **Name:** ____________________

1. Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} se^{-st} & \text{if } 0 \le s \le 1 \text{ and } t \ge 0 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{P}\left\{X \le \frac{1}{4},\, Y \le 6\right\}$.

Answers

1.

$$\mathbb{P}\left\{X \le \frac{1}{4}, Y \le 6\right\} = \int_{s=-\infty}^{1/4}\int_{t=-\infty}^{6} f_{X,Y}(s,t)dsdt = \int_{s=0}^{1/4}\int_{t=0}^{6} se^{-st}dsdt$$
$$= \int_{s=0}^{1/4}\left(1-e^{-6s}\right)ds = \frac{1}{4} - 1 + \exp\left[-\frac{3}{2}\right].$$

Math 461, Section D13, Fall 2010
Quiz 12, December 6 **Name:** ______________________

1. Suppose that $x = 0$. Evaluate the following expressions:
   (a) $\boxed{\text{5 points}}$ $2x$.
   (b) $\boxed{\text{5 points}}$ $x - x$.

Answers

1. (a) $2x = 0$
   (b) $x - x = 0$

Math 461, Section D13, Fall 2010
Exam 1, September 22

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. $\boxed{\text{10 points}}$ (Similar to Question 43 in Chapter 3). There are 4 coins in a box:

   - A two-headed coin
   - Two fair coins
   - A biased coin that comes up heads 75% of the time.

   One of the coins is randomly selected and tossed. If it shows up heads, what is the probability that it is the two-headed coin?

2. $\boxed{\text{30 points}}$ Suppose that

$$\mathbb{P}(A \cap C|B) = .2, \quad \mathbb{P}(A|B) = .6, \quad \text{and} \quad \mathbb{P}(A \cap B) = .4.$$

   Let's do some calculations. Do not confuse $\setminus$, which denotes set subtraction, with $|$, which denotes conditional probability.

   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(B)$.

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(A \setminus C|B)$.

   (c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(A \cap B \cap C)$

3. $\boxed{\text{30 points}}$ Suppose that Alex and Bob are on a team of 20 people. Two committees will be formed; committee A contains 8 people and committee B contains 10 people. The committees can overlap.

   (a) $\boxed{\text{10 points}}$ How many ways can the comittees be formed?

   (b) $\boxed{\text{10 points}}$ What is the probability that Alex and Bob will both be on committee A.

   (c) $\boxed{\text{10 points}}$ What is the probability that Alex and Bob will both be on committee A but they will not both be on committee B?

4. $\boxed{\text{30 points}}$ 4 couples are in a room and to be seated at a dinner table. Each side of the table can accomodate 4 people.

   (a) $\boxed{\text{10 points}}$ How many ways can the people be seated if all the men are on one side, all the women are on the other side?

   (b) $\boxed{\text{10 points}}$ How many ways can the people be seated if each husband and wife are to be seated across from each other?

   (c) $\boxed{\text{10 points}}$ How many ways can the people be seated if each husband and wife are to be seated across from each other, and each person's neighbor is of the opposite gender?

Answers

1.

$$\begin{aligned}\mathbb{P}(C_1|H) &= \frac{\mathbb{P}(C_1\cap H)}{\mathbb{P}(H)} \\ &= \frac{\mathbb{P}(H|C_1)\mathbb{P}(C_1)}{\mathbb{P}(H|C_1)\mathbb{P}(C_1)+\mathbb{P}(H|C_2)\mathbb{P}(C_2)+\mathbb{P}(H|C_3)\mathbb{P}(C_3)+\mathbb{P}(H|C_4)\mathbb{P}(C_4)} \\ &= \frac{1\cdot\frac{1}{4}}{1\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{4}}.\end{aligned}$$

2. (a)

$$\mathbb{P}(B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A|B)} = \frac{.4}{.6} = \frac{2}{3}.$$

(b) $\mathbb{P}(A\setminus C|B) = \mathbb{P}(A|B) - \mathbb{P}(A\cap C|B) = .6 - .2 = .4$

(c) $\mathbb{P}(A\cap B\cap C) = \mathbb{P}(A\cap C|B)\mathbb{P}(B) = .2\times\frac{2}{3}$.

3. (a) $\binom{20}{8}\binom{20}{10}$.

(b)

$$\frac{\binom{18}{6}\binom{20}{10}}{\binom{20}{8}\binom{20}{10}} = \frac{\binom{18}{6}}{\binom{20}{8}}.$$

(c)

$$\begin{aligned}&\mathbb{P}\{\text{Alex and Bob are both on committee A}\} \\ &\qquad - \mathbb{P}\{\text{Alex and Bob are both on committees } A \text{ and } B\} \\ &\qquad = \frac{\binom{18}{6}}{\binom{20}{8}} - \frac{\binom{18}{6}\binom{18}{8}}{\binom{20}{8}\binom{20}{10}}.\end{aligned}$$

4. (a) $4!\times 4!$

(b) $2^4\times 4!$

(c) $2\times 4!$

Math 461, Section D13, Fall 2010
Exam 2, October 29

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
135 points total.

1. $\boxed{\text{35 points}}$ Fix $p \in (0,1)$. Suppose that X is a random variable with $\mathbb{P}\{X=3\}=p$ and $\mathbb{P}\{X=7\}=1-p$.
   (a) $\boxed{\text{8 points}}$ Compute probability mass function p_X of X.
   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X]$.
   (c) $\boxed{\text{5 points}}$ Compute the variance of X.
   (d) $\boxed{\text{7 points}}$ Compute the moment generating $\varphi(\theta) \overset{\text{def}}{=} \mathbb{E}[\exp[\theta X]]$.
   (e) $\boxed{\text{5 points}}$ Compute $\varphi'(\theta)$
   (f) $\boxed{\text{5 points}}$ Compute $\varphi''(\theta)$

2. $\boxed{\text{15 points}}$ Suppose that X is a random variable and $Y=3X+2$. Suppose that $\mathbb{E}[Y]=8$ an $\text{Var}(Y)=27$.
   (a) $\boxed{\text{8 points}}$ Compute $\mathbb{E}[X]$.
   (b) $\boxed{\text{7 points}}$ Compute $\text{Var}(X)$.

3. $\boxed{\text{65 points}}$ Suppose that X is a continuous random variable with density

$$f_X(t) = \begin{cases} \frac{1}{3} & \text{if } 0<t<1 \\ \frac{2}{3} & \text{if } 4<t<5 \\ 0 & \text{else} \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute the cumulative distribution function F_X of X.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$
   (c) $\boxed{\text{10 points}}$ Compute $\text{Var}(X)$
   (d) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[\max\{X,5\}]$.

   Suppose now that we define $Y \overset{\text{def}}{=} \max\{X,5\}$.

   (e) $\boxed{\text{5 points}}$ Graph the function $g(z) \overset{\text{def}}{=} \max\{z,5\}$.
   (f) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{Y \le 4.1\}$
   (g) $\boxed{\text{10 points}}$ Compute cumulative distribution F_Y of Y.

(h) $\boxed{\text{5 points}}$ Is Y a continuous random variable? Why or why not?

4. $\boxed{\text{20 points}}$ (taken from problem 52 in Chapter 4) In a typical month, the average number of plane accidents in the world is 2.5.

   (a) $\boxed{\text{10 points}}$ What is the probability that there is at least 2 accidents in November?

   (b) $\boxed{\text{10 points}}$ What is the probability that there are at least 4 accidents in all of 2011?

Answers

1. (a)
$$p_X(j) = \begin{cases} p & \text{if } j=3 \\ 1-p & \text{if } j=7 \end{cases}$$
   (b) $\mathbb{E}[X] = 3p + 7(1-p) = 7-4p$.
   (c) $\mathbb{E}[X^2] = 9p + 49(1-p) = 49-40p$.
   (d) $\operatorname{Var}(X) = 49 - 40p - (7-4p)^2 = -16p^2 + 16p = 16p(1-p)$.
   (e) $\varphi(\theta) = pe^{3\theta} + (1-p)e^{7\theta}$.
   (f) $\varphi'(\theta) = 3pe^{3\theta} + 7(1-p)e^{7\theta}$.
   (g) $\varphi''(\theta) = 9pe^{3\theta} + 49(1-p)e^{7\theta}$.

2. We have that $X = \frac{1}{3}(Y-2)$.
   (a) $\mathbb{E}[X] = \frac{1}{3}(\mathbb{E}[Y]-2) = 2$.
   (b)
$$\begin{aligned}\operatorname{Var}(X) = \mathbb{E}[(X-2)^2] &= \mathbb{E}\left[\left(\tfrac{1}{3}(Y-2) - \tfrac{1}{3}(\mathbb{E}[Y]-2)\right)^2\right] \\ &= \tfrac{1}{9}\mathbb{E}\left[(Y-\mathbb{E}[Y])^2\right] = \tfrac{1}{9}\operatorname{Var}(Y) = 3.\end{aligned}$$

3. (a)
$$F_X(t) = \int_{s=-\infty}^{t} f_X(s)ds = \begin{cases} 0 & \text{if } t<0 \\ \frac{1}{3}t & \text{if } 0 \le t < 1 \\ \frac{1}{3} & \text{if } 1 \le t < 4 \\ \frac{1}{2} + \frac{2}{3}(t-4) & \text{if } 4 \le t < 5 \\ 1 & \text{if } t \ge 4 \end{cases}$$
   (b)
$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} t f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} tdt + \frac{2}{3}\int_{t=4}^{5} tdt = \frac{1}{3}\times\frac{1}{2} + \frac{2}{3}\times\frac{25-16}{2} = \frac{19}{6}.$$
   (c)
$$\mathbb{E}[X^2] = \int_{t=-\infty}^{\infty} t^2 f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} t^2dt + \frac{2}{3}\int_{t=4}^{5} t^2dt = \frac{1}{3}\times\frac{1}{3} + \frac{2}{3}\times\frac{125-64}{3} = \frac{127}{9};$$
   thus $\operatorname{Var}(X) = \frac{127}{8} - \left(\frac{19}{6}\right)^2$.
   (d) Since $X \le 5$, $\max\{X,5\}$ is identically 5. Thus $\mathbb{E}[\max\{X,5\}] = \mathbb{E}[5] = 5$.
   (e) $\mathbb{P}\{Y \le 4.1\} = \mathbb{P}\{X \le 4.1\} = 0$.

(f) We have that

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 5 \\ 1 & \text{if } t \ge 5 \end{cases}$$

(g) No; F_Y has a jump at 3.

4. (a) The number of accidents X in November should be approximately Poisson(2.5). Thus

$$\mathbb{P}\{X \ge 2\} = 1 - \mathbb{P}\{X \le 1\} = 1 - e^{-2.5}(1 + 2.5).$$

(b) The number of accidents X in 2011 should be approximately Poisson with parameter $\lambda = 2.5 \times 12 = 30$. Thus

$$\mathbb{P}\{X \ge 4\} = 1 - \mathbb{P}\{X \le 3\} = 1 - e^{-30}\left(1 + 30 + \frac{30^2}{2!} + \frac{30^3}{3!}\right).$$

Math 461, Section D13, Fall 2010
Exam 3, November 14 Name: ____________

1. $\boxed{\text{40 points}}$ Let X be a standard Gaussian random variable. Define $Y \stackrel{\text{def}}{=} \frac{1}{X^2}$.

   (a) $\boxed{\text{10 points}}$ Graph the function $g(z) \stackrel{\text{def}}{=} \frac{1}{z^2}$.

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le 9\}$.

   (c) $\boxed{\text{10 points}}$ Compute the cumulative distribution function F_Y of Y in terms of Φ.

   (d) $\boxed{\text{10 points}}$ Compute the density f_Y of Y.

2. $\boxed{\text{10 points}}$ Suppose that the demand for gasoline at a station on a typical day is on average 2000 gallons and has standard deviation 300 gallons. What capacity tank should the owner purchase so that he runs out of gasoline with probability no more than 1% of the time? Use Chebychev's inequality.

3. $\boxed{\text{10 points}}$ Suppose that 90% of ticket-holders on a typical airline flight actually show up. Suppose that an airplane has 200 seats, and the airline sells 210 tickets. What is the probability that some passengers won't get a seat?

4. $\boxed{\text{30 points}}$ Fix $p \in (0,1)$. Suppose that X and Y have joint probability mass function $p_{X,Y}$ where for all integers i and j

$$p_{X,Y}(i,j) = \begin{cases} (1-p)^{i+j-2}p^2 & \text{if } 1 \le i < j \\ (1-p)^{2i-2}p & \text{if } 1 \le i = j \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute $p_Y(6)$.

   (b) $\boxed{\text{10 points}}$ Compute $p_{X|Y}(3|6)$.

   (c) $\boxed{\text{10 points}}$ Compute p_X.

1. (a) $g(0) = \infty$, $g(\pm\infty) = 0$.

   (b)

$$\mathbb{P}\{Y \le 9\} = \mathbb{P}\left\{|X| \ge \frac{1}{3}\right\} = 2\mathbb{P}\left\{X \ge \frac{1}{3}\right\} = 2\left(1 - \mathbb{P}\left\{X < \frac{1}{3}\right\}\right)$$
$$= 2\left(1 - \Phi\left(\frac{1}{3}\right)\right) = 2\,(1 - .6293) = 0.7414$$

   (c) For any $t > 0$,

$$\mathbb{P}\{Y \le t\} = \mathbb{P}\left\{|X| \ge \frac{1}{\sqrt{t}}\right\} = 2\mathbb{P}\left\{X \ge \frac{1}{\sqrt{t}}\right\} = 2\left\{1 - \Phi\left(\frac{1}{\sqrt{t}}\right)\right\}$$

   For $t < 0$, $\mathbb{P}\{Y \le t\} = 0$.

   (d) Differentiate. We get that

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}\right] & \text{if } t > 0 \end{cases}$$

2. Let X be the demand, and let C be the tank capacity. By Chebychev's inequality, for any $C > 2000$ we have that

$$\mathbb{P}\{X \ge C\} = \mathbb{P}\{X - 2000 \ge C - 2000\} \le \left(\frac{300}{C - 2000}\right)^2.$$

   We want that

$$\left(\frac{300}{C - 2000}\right)^2 \le .01$$

   which means that $C \ge 5000$.

3. Let $\xi_n = 1$ if the n-th passenger shows up and zero otherwise. By the central limit theorem,

$$Z \stackrel{\text{def}}{=} \frac{1}{\sqrt{210 \times .9 \times .1}} \sum_{n=1}^{210} (\xi_n - .90)$$

   is approximately Gaussian with mean 0 and variance 1. Thus

$$\mathbb{P}\left\{\sum_{n=1}^{210} \xi_n \ge 200\right\} = \mathbb{P}\left\{\frac{1}{\sqrt{210 \times .9 \times .1}} \sum_{n=1}^{210} (\xi_n - .9) \ge \frac{200 - 210 \times .9}{\sqrt{210 \times .9 \times .1}}\right\} \approx \mathbb{P}\{Z \ge 2.53\}$$
$$= 1 - \mathbb{P}\{Z < 2.53\} = 1 - \Phi(2.53) = 1 - .9943 = .0057.$$

4. (a)

$$\begin{aligned} p_Y(6) &= \sum_i p_{X,Y}(i,6) = \sum_{i=1}^{5}(1-p)^{4+i}p^2 + (1-p)^{10}p \\ &= p^2(1-p)^5\sum_{i'=0}^{4}(1-p)^{i'} + (1-p)^{10}p \\ &= p^2(1-p)^5\frac{1-(1-p)^5}{p} + (1-p)^{10}p = (1-p)^5 p \end{aligned}$$

(b)

$$p_{X|Y}(3|6) = \frac{p_{X,Y}(3,6)}{p_Y(6)} = \frac{(1-p)^7p^2}{(1-p)^5p} = (1-p)^2p.$$

(c) For $j \geq 1$,

$$\begin{aligned} p_Y(j) &= \sum_i p_{X,Y}(i,j) = \sum_{i=1}^{j-1}(1-p)^{i+j-2}p^2 + (1-p)^{2j-2}p \\ &= p^2(1-p)^{j-1}\sum_{i'=0}^{j-2}(1-p)^{i'} + (1-p)^{2j-2}p \\ &= p^2(1-p)^{j-1}\frac{1-(1-p)^{j-1}}{p} + (1-p)^{2j-2}p = (1-p)^{j-1}p \end{aligned}$$

We also have $p_Y(j) = 0$ for $j \leq 0$. Thus for all integers j

$$p_Y(j) = \begin{cases} (1-p)^{j-1}p & \text{if } j \in \{1,2,\dots\} \\ 0 & \text{else} \end{cases}$$

Math 461, Section D13, Fall 2010
Final, December 16

Show all work; partial credit will be given
Calculators should only be used on Questions 1c, 8, 9
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
255 points total.

1. $\boxed{\text{30 points}}$ Suppose that in a typical lottery the probability of winning a price is 1/100. Suppose that we play many lotteries.

   Suppose that we play 500 different lotteries. Let's use the Poisson approximation.

   (a) $\boxed{\text{10 points}}$ What is the probability of winning at least once?

   (b) $\boxed{\text{10 points}}$ What is the probability of winning twice?

   (c) $\boxed{\text{10 points}}$ Suppose now that we really like the rush of winning a lottery. How many lotteries should we play so that the probability of winning at least once is .3 or greater?

2. $\boxed{\text{40 points}}$ Suppose that we have a coin such that $\mathbb{P}\{\text{heads}\} = p$ for some fixed $p \in (0, 1)$. Flip the coin a number of times. Let X_4 be the position of the 4th heads and let X_{10} be the position of the 10th heads.

   (a) $\boxed{\text{10 points}}$ Describe what it means to have $X_4 = 13$ and $X_{10} = 25$.

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X_4 = 13,\ X_{10} = 25\}$.

   (c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X_4 = 13 | X_{10} = 25\}$.

   (d) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X_4 = i | X_{10} = 25\}$ for $i \in \{4, 5 \dots 14\}$

3. $\boxed{\text{30 points}}$ (modelled on question 2.44). Six people are in a room. Three of them are Alex, Beth, and Charlie. They are randomly seated side by side on a bench.

   (a) $\boxed{\text{10 points}}$ What is the probability that Alex and Beth are seated side by side?

   (b) $\boxed{\text{10 points}}$ What is the probability that there are exactly two people between Alex and Beth?

   (c) $\boxed{\text{10 points}}$ What is the probability that Alex, Beth, and Charlie are seated side by side?

4. $\boxed{\text{30 points}}$ Suppose that

$$\mathbb{P}(A) = .6 \qquad \mathbb{P}(B) = .7 \qquad \text{and} \qquad \mathbb{P}(A|B) = .8.$$

   (a) $\boxed{\text{5 points}}$ Are A and B independent?

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(A \cap B)$

   (c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}(A \cup B)$

(d) $\boxed{\text{5 points}}$ Compute $\mathbb{P}((A \cup B)^c \cup (A \cap B))$

5. $\boxed{\text{50 points}}$ Let X be an exponential random variable with parameter 1; i.e., it is continuous with density

$$f_X(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & \text{else} \end{cases}$$

Define

$$\varphi(z) \overset{\text{def}}{=} \begin{cases} 2 & \text{if } z \le 2 \\ z-2 & \text{if } 2 < z \le 7 \\ 5 & \text{if } z > 7 \end{cases}$$

Define $Y \overset{\text{def}}{=} \varphi(X)$.

(a) $\boxed{\text{10 points}}$ Graph φ

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le \frac{1}{2}\}$

(c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le 3\}$.

(d) $\boxed{\text{10 points}}$ Compute the cumulative distribution function of Y.

(e) $\boxed{\text{10 points}}$ What is $\mathbb{P}\{Y = 2\}$?

6. $\boxed{\text{25 points}}$ Suppose that X and Y are two independent random variables with densities

$$f_X(s) = \begin{cases} e^{-s} & \text{if } s \ge 0 \\ 0 & \text{else} \end{cases}$$
$$f_Y(t) = \begin{cases} te^{-t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$.

(a) $\boxed{\text{10 points}}$ What is the joint density $f_{X,Y}$ of X and Y?

(b) $\boxed{\text{10 points}}$ Compute $f_Z(7)$, where f_Z is the density of Z.

(c) $\boxed{\text{5 points}}$ Compute f_Z.

7. $\boxed{\text{30 points}}$ (after question 4.38) Suppose that $\mathbb{E}[X] = 2$ and $\text{Var}(X) = 6$. Compute

(a) $\boxed{\text{10 points}}$ $\mathbb{E}[X^2]$.

(b) $\boxed{\text{10 points}}$ $\mathbb{E}[(3+X)^2]$.

(c) $\boxed{\text{10 points}}$ $\text{Var}(3+2X)$.

8. $\boxed{\text{10 points}}$ Suppose that 55% are in favor of a proposed new law. If we sample 1000 people, what is the probability that we will in fact find that less than 500 are in favor of the law?

9. $\boxed{\text{10 points}}$ Let X be a Gaussian random variable with mean 5 and variance 36. Find the value of c such that $\mathbb{P}\{X \le c\} = .4$.

Answers

1. If we play N lotteries, the number X of times we win is approximately Poisson with parameter $N/100$.

   If $N = 100$, X is Poisson with parameter 5

   (a) $\mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - e^{-5}$.

   (b) $\mathbb{P}\{X = 2\} = \frac{5^2}{2!}e^{-5}$.

   (c) We want that $1 - e^{-N/100} \geq .3$. Thus $N \geq 100 \ln \frac{1}{.7} = 35$.

2. (a) 13th and 25th coins are heads, 3 heads in the first 12, and 5 heads in tosses $\{14, 15 \ldots 24\}$.

   (b) $\mathbb{P}\{X_4 = 13,\, X_{10} = 25\} = \binom{12}{3}\binom{11}{5}p^{10}(1-p)^{15}$.

   (c) $\mathbb{P}\{X_{10} = 25\} = \binom{24}{9}(1-p)^{10}(1-p)^{15}$, so

   $$\mathbb{P}\{X_4 = 13 | X_{10} = 20\} = \frac{\binom{12}{3}\binom{11}{5}}{\binom{24}{9}}.$$

   (d)

   $$\mathbb{P}\{X_4 = i | X_{10} = 25\} = \frac{\binom{i-1}{3}\binom{24-i}{5}}{\binom{24}{9}}.$$

3. (a) $5! \times 2/6! = \frac{2}{6} = \frac{1}{3}$.

   (b) $3 \times 2 \times 4!/6! = \frac{6}{30} = \frac{1}{5}$.

   (c) $4! \times 3!/6! = \frac{3 \times 2}{6 \times 5} = \frac{1}{5}$.

4. (a) No, since $\mathbb{P}(A|B) = .6 \neq .2 = \mathbb{P}(A)$.

   (b) $\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B) = .8 \times .7 = .56$.

   (c) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = .6 + .7 - .56 = .74$.

   (d) $\mathbb{P}((A \cup B)^c \cup (A \cap B)) = \mathbb{P}((A \cup B)^c) + \mathbb{P}(A \cap B) = 1 - \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B) = 1 - .74 + .56 = .82$.

5. (a) φ is flat at 2 to the left of 2, jumps down to 0 at 2, increases at 45° from 2 to 7, and then stays flat.

   (b)

   $$\mathbb{P}\{Y \leq \tfrac{1}{2}\} = \mathbb{P}\{2 \leq X \leq 2.5\} = \int_{s=2}^{2.5} f_X(s)ds = e^{-2} - 2^{-2.5}.$$

   (c)

   $$\mathbb{P}\{Y \leq 3\} = \mathbb{P}\{X \leq 5\} = \int_{s=-\infty}^{5} f_X(s)ds = 1 - e^{-5}.$$

(d)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathbb{P}\{2 \le X \le 2+t\} & \text{if } 0 \le t < 2 \\ \mathbb{P}\{X \le 2+t\} & \text{if } 2 \le t < 7 \\ 1 & \text{if } t \ge 7 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ e^{-2} - e^{-2-t} & \text{if } 0 \le t < 2 \\ 1 - e^{-2-t} & \text{if } 2 \le t < 5 \\ 1 & \text{if } t \ge 5 \end{cases}$$

(e)

$$\mathbb{P}\{Y = 2\} = \mathbb{P}\{X \le 2\} + \mathbb{P}\{X = 4\} = 1 - e^{-2}.$$

6. (a)

$$f_{X,Y}(s,t) = \begin{cases} te^{-s-t} & \text{if } s \ge 0 \text{ and } t \ge 0 \\ 0 & \text{else} \end{cases}$$

(b)

$$f_Z(7) = \int_{t'=-\infty}^{\infty} f_X(7-t') f_Y(t') dt' = \int_{t'=0}^{7} e^{-(7-t')} t' e^{-t'} dt' = e^{-7} \int_{t'=0}^{7} t' dt' = \frac{7^2}{2} e^{-7}.$$

(c) $f_Z(t) = 0$ if $t < 0$. If $t > 0$,

$$f_Z(t) = \int_{t'=-\infty}^{\infty} f_X(t-t') f_Y(t') dt' = \int_{t'=0}^{t} e^{-(t-t')} t' e^{-t'} dt' = e^{-t} \int_{t'=0}^{t} t' dt' = \frac{t^2}{2} e^{-t}.$$

Thus

$$f_Z(t) = \begin{cases} \frac{t^2}{2} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

7. (a) $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 6 + 4 = 10.$
   (b) $\mathbb{E}[(3+X)^2] = 9 + 6\mathbb{E}[X] + \mathbb{E}[X^2] = 9 + 6 \times 2 + 10 = 31.$
   (c) $\text{Var}(3+X) = \mathbb{E}[(3+X)^2] - \mathbb{E}[3+X]^2 = 31 - (3+2)^2 = 31 - 25 = 6.$

8. Let $\xi_n = 1$ if the n-th student is in favor of the law and zero otherwise. Then $\mathbb{P}\{\xi_n = 1\} = .55$. By the central limit theorem,

$$Z \stackrel{\text{def}}{=} \frac{1}{\sqrt{1000 \times .55 \times .45}} \sum_{n=1}^{1000} (\xi_n - .5)$$

is approximately Gaussian with mean 0 and variance 1. If N is a standard Gaussian, we thus have that

$$\begin{aligned} \mathbb{P}\left\{\sum_{n=1}^{1000} \xi_n \le 500\right\} &= \mathbb{P}\left\{\frac{1}{\sqrt{1000 \times .55 \times .45}} \sum_{n=1}^{1000} (\xi_n - .55) \le \frac{500 - 1000 \times .55}{\sqrt{1000 \times .55 \times .45}}\right\} \\ &\approx \mathbb{P}\{N \le -3.17\} \\ &= \mathbb{P}\{N \ge 3.17\} = 1 - \Phi(3.17) = 1 - .9992 = .0002. \end{aligned}$$

9. We can write $X = 6Z + 5$, where Z is a standard normal. Thus we want

$$.4 = \mathbb{P}\{X \le c\} = \mathbb{P}\{6Z + 5 \le c\} = \mathbb{P}\left\{Z \le \frac{c-5}{6}\right\} = \Phi\left(\frac{c-5}{6}\right).$$

Since $\Phi(x) \ge .5$ when $x \ge 0$, we know that $\frac{1}{6}(c-5) < 0$. We thus calculate that

$$.4 = \mathbb{P}\left\{Z \le \frac{c-5}{6}\right\} = \mathbb{P}\left\{Z \ge -\frac{c-5}{6}\right\} = 1 - \Phi\left(\frac{5-c}{6}\right)$$

Hence we want that

$$\Phi\left(\frac{5-c}{6}\right) = 1 - .4 = .6.$$

Hence

$$\frac{5-c}{6} = .26$$

so $c = 4.844$.