Math 461, Section C13, Fall 2010
Quiz 1, August 27 **Name:** ____________________

1. $\boxed{\text{10 points}}$ (taken from problem 20 on p. 17) Angus has 10 friends, two of whom are named Brittany and Carl. Angus is planning to invite 5 of the friends to a party.
   (a) $\boxed{\text{5 points}}$ How many ways can the friends be invited if Brittany and Carl won't come together?
   (b) $\boxed{\text{5 points}}$ How many ways can the friends be invited if Brittany and Carl will only attend together?

Answers

1. (a)

$$\binom{10}{5} - \binom{8}{3}.$$

(b)

$$\binom{8}{3} + \binom{8}{5}.$$

Math 461, Section C13, Fall 2010
Quiz 2, September 1 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that we have 8 women, and 15 men. One of the women is named Alice, and one of the men is named Bob. Suppose that we form a committee of 4 women and 6 men. What is the probability that Alice and Bob do not serve together? (To make sense of this, suppose that Alice and Bob do not get along. If they are not on the committee together, the committee will be a peaceful one. We are interested in the probability that the committee will be a peaceful one).

Answers

1.

$$1-\frac{\binom{7}{3}\binom{14}{5}}{\binom{8}{4}\binom{15}{6}}.$$

Math 461, Section C13, Fall 2010
Quiz 3, September 10 **Name:** ______________________

1. $\boxed{\text{10 points}}$ (taken from question 3.6 on p. 102) A box contains 9 white and 7 red balls. Sample 6 times without replacement. What is the probability that the first and fourth balls are white given that you picked exactly 4 white balls?

Answers

1.

$$\frac{(9)_4(7)_2\binom{4}{2}}{(9)_4(7)_2\binom{6}{4}} = \frac{\binom{4}{2}}{\binom{6}{4}}$$

Math 461, Section C13, Fall 2010
Quiz 4, September 15 **Name:** ____________

1. $\boxed{\text{10 points}}$ (taken from question 3.35 on p. 104) It's your birthday! Your mother or father has hidden a present. With probability .6, the present was hidden by your father. If your father hides a present, it he hides it upstairs 70% of the time, while if your mother hides a present, it is equally likely to be upstairs or downstairs.

   (a) $\boxed{\text{5 points}}$ What is the probability that the present is upstairs?

   (b) $\boxed{\text{5 points}}$ If the present is downstairs, what is the probability that it was hidden by your father?

Answers

1.

$$U = \{\text{upstairs}\}, \qquad F = \{\text{hidden by father}\}.$$

(a)

$$\mathbb{P}(U) = \mathbb{P}(U|F)\mathbb{P}(F) + \mathbb{P}(U|F^c)\mathbb{P}(F^c) = .7 \times .6 + .5 \times .4.$$

(b)

$$\mathbb{P}(F|U^c) = \frac{\mathbb{P}(U^c|F)\mathbb{P}(F)}{\mathbb{P}(U^c|F)\mathbb{P}(F) + \mathbb{P}(U^c|F^c)\mathbb{P}(F^c)} = \frac{.3 \times .6}{.3 \times .6 + .5 \times .4}.$$

Math 461, Section C13, Fall 2010
Quiz 5, September 29 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that we toss 10 independent coins, and that $\mathbb{P}\{\text{heads}\} = \frac{4}{5}$ for each coin. Let $X = 1$ if we get exactly 3 heads, and $X = 0$ otherwise. Compute the probability mass function p_X of X.

Answers

1.
$$p_X(j) = \begin{cases} \binom{10}{3} \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^7 & \text{if } j = 1 \\ 1 - \binom{10}{3} \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^7 & \text{if } j = 0 \\ 0 & \text{else} \end{cases}$$

Math 461, Section C13, Fall 2010
Quiz 6, October 1 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that we toss a sequence of coins, where $\mathbb{P}\{\text{heads}\} = p$ for some $p \in (0,1)$. Let X be the position of the first heads. Define the function $f(z) \overset{\text{def}}{=} \min\{z, 10\}$, and define $Y \overset{\text{def}}{=} \min\{X, 10\}$.
   (a) $\boxed{\text{3 points}}$ Graph f.
   (b) $\boxed{\text{7 points}}$ Compute the probability mass function of Y.

Answers

(a) $f(z) = z$ for $z \le 10$, and $f(z) = 10$ for $z \ge 10$.

(b)

$$p_Y(j) = \begin{cases} (1-p)^{j-1}p & \text{if } j \in \{1, 2 \dots 10\} \\ (1-p)^9 & \text{if } j = 10 \\ 0 & \text{else} \end{cases}$$

Math 461, Section C13, Fall 2010
Quiz 7, October 13 Name: ____________________

1. $\boxed{\text{10 points}}$ Let X be a random variable such that

$$\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[\exp\left[\theta X\right]\right] = \begin{cases} \frac{5}{5-\theta} & \text{if } \theta < 5 \\ \infty & \text{else} \end{cases}$$

   (a) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X]$.

   (b) $\boxed{\text{5 points}}$ Compute the variance of X.

For $\theta < 5$, we have that

$$\varphi'_X(\theta) = \frac{5}{(5-\theta)^2} \qquad \text{and} \qquad \varphi''_X(\theta) = \frac{10}{(5-\theta)^3}.$$

(a) $\mathbb{E}[X] = \varphi'_X(0) = \frac{1}{5}$.

(b) $\mathrm{V}(X) = \varphi''_X(\theta) - (\varphi'_X(\theta))^2 = \frac{10}{5^3} - \left(\frac{1}{5}\right)^2 = \frac{1}{5^2}$.

Math 461, Section C13, Fall 2010
Quiz 8, October 13 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Let X be a random variable with cumulative distribution function

$$F_X(t) = \mathbb{P}\{X \le t\} = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{4}t & \text{if } 0 \le t < 1 \\ \frac{1}{4} + \frac{3}{8}(t-1) & \text{if } 1 \le t < 3 \\ 1 & \text{if } t \ge 3. \end{cases}$$

Compute the density function of X.

Answers

$$f_X(t) = F_X'(t) = \begin{cases} \frac{1}{4} & \text{if } 0 < t < 1 \\ \frac{3}{8} & \text{if } 1 < t < 3 \\ 0 & \text{else} \end{cases}$$

Math 461, Section C13, Fall 2010
Quiz 9, November 12 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Let X be a Gaussian random variable with mean 5 and variance 36. Find the value of c such that $\mathbb{P}\{X \le c\} = .4$.

We can write $X = 6Z + 5$, where Z is a standard normal. Thus we want

$$.4 = \mathbb{P}\{X \le c\} = \mathbb{P}\{6Z + 5 \le c\} = \mathbb{P}\left\{Z \le \frac{c-5}{6}\right\} = \mathbb{P}\left\{Z \le \frac{c-5}{6}\right\} = \Phi\left(\frac{c-5}{6}\right).$$

Since $\Phi(x) \ge .5$ when $x \ge 0$, we know that $\frac{1}{6}(c-5) < 0$. We thus calculate that

$$.4 = \mathbb{P}\left\{Z \le \frac{c-5}{6}\right\} = \mathbb{P}\left\{Z \ge -c\frac{c-5}{6}\right\} = 1 - \Phi\left(-\frac{c-5}{6}\right)$$

Hence we want that

$$\Phi\left(-\frac{c-5}{6}\right) = 1 - .4 = .6.$$

Hence

$$-\frac{c-5}{6} = .26$$

so

$$c = 5 - .6. \times .26 = 4.844.$$

Math 461, Section C13, Fall 2010
Quiz 10, November 15 **Name:** ______________________

1. $\boxed{\text{10 points}}$ Fix $p \in (0,1)$. Suppose that X and Y have joint density $p_{X,Y}$ where for all integers i and j

$$p_{X,Y}(i,j) = \begin{cases} (1-p)^{i+j-2}p^2 & \text{if } 1 \le i < j \\ (1-p)^{2i-2}p & \text{if } 1 \le i = j \end{cases}$$

Compute $p_X(8)$.

1. We have that

$$
\begin{aligned}
p_X(8) = \sum_j p_{X,Y}(8,j) = p_{X,Y}(8,8) + \sum_{j=9}^\infty p_{X,Y}(8,j) = (1-p)^{2\times 8-2}p + \sum_{j=9}^\infty (1-p)^{6+j}p^2 \\
= (1-p)^{14}p + (1-p)^6 p^2 \sum_{j=9}^\infty (1-p)^j = (1-p)^{14}p + (1-p)^6p^2\frac{(1-p)^9}{p} \\
= (1-p)^{14}p + (1-p)^{15}p = (1-p)^{14}p\left\{2-p\right\}.
\end{aligned}
$$

Math 461, Section C13, Fall 2010
Quiz 11, December 3 **Name:** ______________________

1. Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} se^{-st} & \text{if } 0 \le s \le 1 \text{ and } t \ge 0 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{P}\left\{X \le \frac{3}{4}, Y \le 5\right\}$.

Answers

1.

$$\mathbb{P}\left\{X \le \frac{3}{4},\, Y \le 5\right\} = \int_{s=-\infty}^{3/4}\int_{t=-\infty}^{5} f_{X,Y}(s,t)\,ds\,dt = \int_{s=0}^{3/4}\int_{t=0}^{5} se^{-st}\,ds\,dt$$
$$= \int_{s=0}^{3/4}\left(1-e^{-5s}\right)ds = \frac{3}{4} - 1 + \exp\left[-\frac{15}{4}\right].$$

Math 461, Section C13, Fall 2010
Quiz 12, December 6 **Name:** ______________________

1. Suppose that $x = 0$. Evaluate the following expressions:
   (a) $\boxed{\text{5 points}}$ x^2.
   (b) $\boxed{\text{5 points}}$ x^3.

Answers

1. (a) $x^2 = 0$

   (b) $x^3 = 0$

Math 461, Section C13, Fall 2010
Exam 1, September 22

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. $\boxed{\text{20 points}}$ (Question 37 in Chapter 2) An instructor gives her class a set of 10 study problems to figure out in preparation for an exam. The exam will consist of 5 of these problems and 5 other problems. Sammy Student has figured out 7 of the study problems. Also, he knows about 80% of the material in the chapter; i.e., for a typical random question from the material, he will get it right 80% of the time.
   (a) $\boxed{\text{10 points}}$ What is the probability that he will get a perfect score on the exam?
   (b) $\boxed{\text{10 points}}$ What is the probability that he will 90% on the exam?

2. $\boxed{\text{40 points}}$ Suppose that we have want to arrange our textbooks on a shelf. Suppose that we have
   - Language textbooks in Spanish, Mandarin Chinese, and Urdu (3 total)
   - Novels (i.e., literature textbooks) by José Saramago, Djuna Barnes, Norman Mailer, and Maya Angelou (4 total)
   - Mathematics textbooks in Probability, Calculus, Differential Equations, Statistics, and Number Theory (5 total)
   - Science textbooks in Physics, Biology, Cosmology, Geology, Organic Chemistry, and Inorganic Chemistry (6 total)

   Suppose that we want to keep the language textbooks together, the literature textbooks together, the mathematics textbooks together, and the science textbooks together.
   (a) $\boxed{\text{10 points}}$ How many ways are there to arrange the books?
   (b) $\boxed{\text{10 points}}$ What is the probability that José Saramago will be put next to the calculus book?

   Suppose now that we need to sell 2 books
   (c) $\boxed{\text{10 points}}$ How many ways can we do this if both books are from the same field (i.e., languages, literature, math, or science)?
   (d) $\boxed{\text{10 points}}$ How many ways can we do this if both books are from different fields (i.e., languages, literature, math, or science)?

3. $\boxed{\text{20 points}}$ Players A and B alternate rolling a pair of dice (A starts) stopping when A wins by rolling a 7 or B wins by rolling a 5.
   (a) $\boxed{\text{10 points}}$ Compute the probability that B wins on her 3-rd toss.
   (b) $\boxed{\text{10 points}}$ Compute the probability that B wins.

4. $\boxed{\text{20 points}}$ (Like example 3m). Suppose that a box contains 100 flashlights. Twenty of the flashlights are type A, 30 are of type B and the remaining 50 are of type C. Flashlights of type A work 90% of the time, flashlights of type B work 70% of the time, and flashlights of type C work 60% of the time.

   (a) $\boxed{\text{10 points}}$ What is the probability that a randomly selected flashlight will work?

   (b) $\boxed{\text{10 points}}$ If a randomly selected flashlight works, what is the probability that it is either type A or B?

1. (a)

$$(0.8)^5 \frac{\binom{7}{5}}{\binom{10}{5}}$$

(b)

$$(0.8)^4(0.2)\frac{\binom{7}{5}}{\binom{10}{5}} + (0.8)^5\frac{\binom{7}{4}\cdot 3}{\binom{15}{5}}.$$

2. (a) $3! \times 4! \times 5! \times 6! \times 4!$

(b)

$$\frac{3! \times 4! \times 3! \times 6! \times 2 \times 3!}{3! \times 4! \times 5! \times 6! \times 4!} = \frac{3! \times 4! \times 2 \times 3!}{4! \times 5! \times 4!} = \frac{2}{4 \times 5 \times 4} = \frac{1}{40}$$

Explanation of numerator: 3! ways to order non-Saramago lit books, 4! ways to order non-calculus math books. 3! ways to order language books. 6! ways to order science books. 2 for lit then math as opposed to math then lit, then 3! ways to order subjects, where lit-math is grouped together.

(c)

$$\binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2}.$$

(d)

$$3 \times 4 + 3 \times 5 + 3 \times 6 + 4 \times 5 + 4 \times 6 + 5 \times 6.$$

3. Note that for a single toss of the dice

$$\mathbb{P}\{7\} = \frac{6}{36} \qquad \text{and} \qquad \mathbb{P}\{5\} = \frac{4}{36}.$$

(a) Call the set in question B_3. Then

$$\mathbb{P}(B_3) = \left(\frac{32}{36}\right)^2 \frac{4}{36}\left(\frac{30}{36}\right)^3.$$

(b)

$$\begin{aligned}\mathbb{P}\{B \text{ wins}\} &= \sum_{n=1}^{\infty}\left(\frac{32}{36}\right)^{n-1}\frac{4}{36}\left(\frac{30}{36}\right)^n = \frac{4}{36}\frac{30}{36}\sum_{n=1}^{\infty}\left(\frac{32}{36}\frac{30}{36}\right)^{n-1} \\ &= \frac{4}{36}\frac{30}{36}\sum_{j=0}^{\infty}\left(\frac{32}{36}\frac{30}{36}\right)^{j} = \frac{(4/36)(30/36)}{(1-(32/36)(30/36))} = \frac{4 \times 30}{36^2 - 32 \times 30}.\end{aligned}$$

4. (a)

$$\mathbb{P}(W) = \mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|B)\mathbb{P}(B) + \mathbb{P}(W|C)\mathbb{P}(C) = 0.9 \times 0.2 + 0.7 \times .3 + 0.6 \times 0.5.$$

(b)

$$\mathbb{P}(A\cup B|W) = \frac{\mathbb{P}((A\cup B)\cap W))}{\mathbb{P}(W)} = \frac{\mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|B)\mathbb{P}(B)}{\mathbb{P}(W|A)\mathbb{P}(A) + \mathbb{P}(W|B)\mathbb{P}(B) + \mathbb{P}(W|C)\mathbb{P}(C)}$$
$$= \frac{0.9\times 0.2 + 0.7\times .3}{0.9\times 0.2 + 0.7\times .3 + 0.6\times 0.5}.$$

Math 461, Section C13, Fall 2010
Exam 2, October 29

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
150 points total.

1. $\boxed{\text{35 points}}$ Fix $p \in (0,1)$. Suppose that X is a random variable with $\mathbb{P}\{X=3\} = p$ and $\mathbb{P}\{X=7\} = 1-p$.
   (a) $\boxed{\text{8 points}}$ Compute probability mass function p_X of X.
   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X]$.
   (c) $\boxed{\text{5 points}}$ Compute the variance of X.
   (d) $\boxed{\text{7 points}}$ Compute the moment generating $\phi(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]]$.
   (e) $\boxed{\text{5 points}}$ Compute $\phi'(\theta)$
   (f) $\boxed{\text{5 points}}$ Compute $\phi''(\theta)$

2. $\boxed{\text{30 points}}$ (based on question 65 in chapter 4) Each of 600 soldiers independently has a disease with probability $1/10^3$. A blood test can detect this disease.
   (a) $\boxed{\text{10 points}}$ Let X be the number of soldiers who have the disease. What sort of random variable (with parameters) is an appropriate model for X?

   The first step is to pool all of the blood and test the combined blood
   (b) $\boxed{\text{10 points}}$ What is the probability that the pool of blood tests positive for the disease?
   (c) $\boxed{\text{10 points}}$ Conditional on the pool of blood testing positive, what is the probability that more than one soldier is infected?

3. $\boxed{\text{65 points}}$ Suppose that X is a continuous random variable with density

$$f_X(t) = \begin{cases} \frac{1}{3} & \text{if } 0 < t < 1 \\ \frac{2}{3} & \text{if } 4 < t < 5 \\ 0 & \text{else} \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute the cumulative distribution function F_X of X.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$
   (c) $\boxed{\text{10 points}}$ Compute $\text{Var}(X)$
   (d) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[\max\{X,5\}]$.

   Suppose now that we define $Y \stackrel{\text{def}}{=} \max\{X,5\}$.

(e) $\boxed{\text{5 points}}$ Graph the function $g(z) \stackrel{\text{def}}{=} \max\{z, 5\}$.

(f) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{Y \le 4.1\}$

(g) $\boxed{\text{10 points}}$ Compute cumulative distribution F_Y of Y.

(h) $\boxed{\text{5 points}}$ Is Y a continuous random variable? Why or why not?

4. $\boxed{\text{20 points}}$ Suppose that X is a random variable with

$$\phi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[\exp\left[\theta X\right]\right] = \begin{cases} \left(\frac{3}{3-\theta}\right)^5 & \text{if } \theta < 3 \\ \infty & \text{else} \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.

(b) $\boxed{\text{10 points}}$ Suppose that $Y \stackrel{\text{def}}{=} 2X + 4$. Compute $\phi_Y(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[\exp\left[\theta Y\right]\right]$

1. (a)
$$p_X(j) = \begin{cases} p & \text{if } j = 3 \\ 1-p & \text{if } j = 7 \end{cases}$$
(b) $\mathbb{E}[X] = 3p + 7(1-p) = 7 - 4p$.
(c) $\mathbb{E}[X^2] = 9p + 49(1-p) = 49 - 40p$.
(d) $\mathrm{Var}(X) = 49 - 40p - (7-4p)^2 = -16p^2 + 16p = 16p(1-p)$.
(e) $\phi(\theta) = pe^{3\theta} + (1-p)e^{7\theta}$.
(f) $\phi'(\theta) = 3pe^{3\theta} + 7(1-p)e^{7\theta}$.
(g) $\phi''(\theta) = 9pe^{3\theta} + 49(1-p)e^{7\theta}$.

2. (a) Poisson with parameter $\lambda = 600 \times 10^{-3} = .6$.
(b) $\mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - e^{-.6}$.
(c)
$$\mathbb{P}\{X > 1 | X \geq 1\} = \frac{\mathbb{P}\{X > 1\}}{\mathbb{P}\{X \geq 1\}} = \frac{1 - e^{-.6} - e^{-.6}.6}{1 - e^{-.6}} = 1 - \frac{.6}{e^{.6} - 1}.$$

3. (a)
$$F_X(t) = \int_{s=-\infty}^{t} f_X(s)ds = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{3}t & \text{if } 0 \leq t < 1 \\ \frac{1}{3} & \text{if } 1 \leq t < 4 \\ \frac{1}{2} + \frac{2}{3}(t-4) & \text{if } 4 \leq t < 5 \\ 1 & \text{if } t \geq 4 \end{cases}$$
(b)
$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} t f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} t dt + \frac{2}{3}\int_{t=4}^{5} t dt = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{25-16}{2} = \frac{19}{6}.$$
(c)
$$\mathbb{E}[X^2] = \int_{t=-\infty}^{\infty} t^2 f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} t^2 dt + \frac{2}{3}\int_{t=4}^{5} t^2 dt = \frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{125-64}{3} = \frac{127}{9};$$
thus $\mathrm{Var}(X) = \frac{127}{8} - \left(\frac{19}{6}\right)^2$.
(d) Since $X \leq 5$, $\max\{X, 5\}$ is identically 5. Thus $\mathbb{E}[\max\{X, 5\}] = \mathbb{E}[5] = 5$.
(e) $\mathbb{P}\{Y \leq 4.1\} = \mathbb{P}\{X \leq 4.1\} = 0$.
(f) We have that
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 5 \\ 1 & \text{if } t \geq 5 \end{cases}$$

(g) No; F_Y has a jump at 3.

4. (a)

$$\mathbb{E}[X] = \phi'_X(0) = \frac{5}{3}.$$

(b)

$$\phi'_Y(\theta) = \mathbb{E}\left[\exp\left[\theta(2X+4)\right]\right] = e^{4\theta}\mathbb{E}\left[\exp\left[(2\theta)X\right]\right] = \begin{cases} e^{4\theta}\left(\frac{3}{3-2\theta}\right)^5 & \text{if } \theta < \frac{3}{2} \\ \infty & \text{else} \end{cases}$$

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Exam 3, November 14 Name: ______________

Throughout,

$$\Phi(t) \stackrel{\text{def}}{=} \int_{s=-\infty}^{t} \frac{\exp\left[-\frac{s^2}{2}\right]}{\sqrt{2\pi}} ds$$

for all $t \in \mathbb{R}$.

1. $\boxed{\text{10 points}}$ (from 5.28 of Ch. 5). 30% of the population is left-handed. Approximate the probability that there are ar least 70 left-handed people in a school of 200 students.

2. $\boxed{\text{30 points}}$ Let X be a standard Gaussian random variable. Define $Y \stackrel{\text{def}}{=} X^2$.
   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le 9\}$.
   (b) $\boxed{\text{10 points}}$ Compute the cumulative distribution function F_Y of Y in terms of Φ.
   (c) $\boxed{\text{10 points}}$ Compute the density f_Y of Y.

3. $\boxed{\text{10 points}}$ Suppose that the demand for gasoline at a station on a typical day is on average 2000 gallons and has standard deviation 300 gallons. What capacity tank should the owner purchase so that he runs out of gasoline with probability no more than 1% of the time? Use Chebychev's inequality.

4. $\boxed{\text{30 points}}$ Fix $p \in (0,1)$. Suppose that X and Y have joint probability mass function $p_{X,Y}$ where for all integers i and j

$$p_{X,Y}(i,j) = \begin{cases} (1-p)^{i+j-2}p^2 & \text{if } 1 \le i < j \\ (1-p)^{2i-2}p & \text{if } 1 \le i = j \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute $p_Y(6)$.
   (b) $\boxed{\text{10 points}}$ Compute $p_{X|Y}(3|6)$.
   (c) $\boxed{\text{10 points}}$ Compute p_X.

5. $\boxed{\text{20 points}}$ Let X be Gaussian with mean 0 and variance 1. Define $Y \stackrel{\text{def}}{=} \Phi(X)$.
   (a) $\boxed{\text{10 points}}$ Fix x^* such that $\Phi(x^*) = .9608$.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le .9608\}$.

Answers

1. Let $\xi_n = 1$ if the n-th student is left-handed, and zero otherwise. By the central limit theorem,
$$Z \stackrel{\text{def}}{=} \frac{1}{\sqrt{200 \times .2 \times .8}} \sum_{n=1}^{200} (\xi_n - .3)$$
is approximately Gaussian with mean 0 and variance 1. Thus
$$\begin{aligned}\mathbb{P}\left\{\sum_{n=1}^{200} \xi_n \geq 70\right\} &= \mathbb{P}\left\{\frac{1}{\sqrt{200 \times .3 \times .7}} \sum_{n=1}^{200} (\xi_n - .3) \geq \frac{70 - 200 \times .3}{\sqrt{200 \times .3 \times .7}}\right\} \approx \mathbb{P}\{Z \geq 1.54\} \\ &= 1 - \mathbb{P}\{Z < 1.54\} = 1 - \Phi(1.54) = 1 - .9382 = .0618.\end{aligned}$$

2. (a)
$$\begin{aligned}\mathbb{P}\{Y \leq 9\} = \mathbb{P}\{X^2 \leq 9\} = \mathbb{P}\{-3 \leq X \leq 3\} &= \Phi(3) - \Phi(-3-) \\ &= \Phi(3) - \{1 - \Phi(3)\} = 2\Phi(3) - 1 = 2 \times .9987 - 1 = .9974.\end{aligned}$$

   (b) For any $t > 0$,
$$\begin{aligned}\mathbb{P}\{Y \leq t\} = \mathbb{P}\{X^2 \leq t\} = \mathbb{P}\{-\sqrt{t} \leq X \leq \sqrt{t}\} &= \Phi(\sqrt{t}) - \Phi(-\sqrt{t}-) \\ &= \Phi(\sqrt{t}) - \left\{1 - \Phi(\sqrt{t})\right\} = 2\Phi(\sqrt{t}) - 1.\end{aligned}$$
   Thus
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2\Phi(\sqrt{t}) - 1 & \text{if } t \geq 0 \end{cases}$$

   (c) Differentiate. We get that
$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{t}{2}\right] & \text{if } t > 0 \end{cases}$$

3. Let X be the demand, and let C be the tank capacity. By Chebychev's inequality, for any $C > 2000$ we have that
$$\mathbb{P}\{X \geq C\} = \mathbb{P}\{X - 2000 \geq C - 2000\} \leq \left(\frac{300}{C - 2000}\right)^2.$$
We want that
$$\left(\frac{300}{C - 2000}\right)^2 \leq .01$$
which means that $C \geq 5000$.

4. (a)

$$\begin{aligned}
p_Y(6) &= \sum_i p_{X,Y}(i,6) = \sum_{i=1}^{5}(1-p)^{4+i}p^2 + (1-p)^{10}p \\
&= p^2(1-p)^5\sum_{i'=0}^{4}(1-p)^{i'} + (1-p)^{10}p \\
&= p^2(1-p)^5\frac{1-(1-p)^5}{p} + (1-p)^{10}p = (1-p)^5 p
\end{aligned}$$

(b)

$$p_{X|Y}(3|6) = \frac{p_{X,Y}(3,6)}{p_Y(6)} = \frac{(1-p)^7p^2}{(1-p)^5p} = (1-p)^2p.$$

(c) For $j \ge 1$,

$$\begin{aligned}
p_Y(j) &= \sum_i p_{X,Y}(i,j) = \sum_{i=1}^{j-1}(1-p)^{i+j-2}p^2 + (1-p)^{2j-2}p \\
&= p^2(1-p)^{j-1}\sum_{i'=0}^{j-2}(1-p)^{i'} + (1-p)^{2j-2}p \\
&= p^2(1-p)^{j-1}\frac{1-(1-p)^{j-1}}{p} + (1-p)^{2j-2}p = (1-p)^{j-1}p
\end{aligned}$$

We also have $p_Y(j)=0$ for $j \le 0$. Thus for all integers j

$$p_Y(j) = \begin{cases} (1-p)^{j-1}p & \text{if } j \in \{1,2,\dots\} \\ 0 & \text{else} \end{cases}$$

5. (a) $x^* = 1.76$.

(b)

$$\mathbb{P}\{Y \le .9608\} = \mathbb{P}\{\Phi(X) \le .9608\} = \mathbb{P}\{X \le 1.76\} = \Phi(1.76) = .9608.$$

Math 461, Section C13, Fall 2010
Final, December 13

Show all work; partial credit will be given
Calculators should only be used on Question 4
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
250 points total.

1. $\boxed{\text{40 points}}$ A math department of 30 people contains three brothers, Alex, Alfred, and Arnold. We need to form an Honors committee of 7 people.
   (a) $\boxed{\text{10 points}}$ How many ways can the committee be formed?
   (b) $\boxed{\text{10 points}}$ What is the probability that Alex will be on the committee?

   Due to nepotism issues, we are concerned with how many of the brothers are on the committee.
   (c) $\boxed{\text{10 points}}$ What is the probability that all three brothers will be on the committee?
   (d) $\boxed{\text{10 points}}$ Conditional on Alex being on the committee, what is the probability that all three brothers are on the committee?

2. $\boxed{\text{10 points}}$ (modelled after question 3.15 in Chapter 3). An ectopic pregnancy is twice as much to occur when the pregnant woman is a smoker as when she is a nonsmoker. Suppose that 30% of women of childbearing age are smokers. What percentage of women having an ectopic pregnancy are smokers (hint: let p be the conditional probability that an ectopic pregnancy occurs when the woman is a non-smoker)?

3. $\boxed{\text{30 points}}$ (modelled on question 2.44). Six people are in a room, and three of them are names Alex, Beth, and Charlie. They are randomly seated side by side on a bench.
   (a) $\boxed{\text{10 points}}$ What is the probability that Alex and Beth are seated side by side?
   (b) $\boxed{\text{10 points}}$ What is the probability that there are exactly two people between Alex and Beth?
   (c) $\boxed{\text{10 points}}$ What is the probability that Alex, Beth, and Charlie are seated side by side?

4. $\boxed{\text{30 points}}$ (based on question 2.42). A pair of dice is repeatedly thrown.
   (a) $\boxed{\text{10 points}}$ Compute the probability that a double 6 is thrown at least once in the first 5 times.
   (b) $\boxed{\text{10 points}}$ Compute the probability that a double 6 is thrown at least once in the first N times
   (c) $\boxed{\text{10 points}}$ Suppose that we want to see a double 6. How many times N do we need to throw the pair of dice so that the probability of throwing a double 6 at least once is at leat .9?

5. $\boxed{\text{40 points}}$ Suppose that X has cumulative distribution function such that

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{4} & \text{if } 0 \le t < 1 \\ \frac{1}{2} + \frac{t-1}{4} & \text{if } 1 \le t < 2 \\ \frac{11}{12} & \text{if } 2 < t < 3 \\ 1 & \text{if } t \ge 3 \end{cases}$$

(a) $\boxed{\text{10 points}}$ Note that I haven't specified the value of F_X at $t = 2$. What must $F_X(2)$ be?

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X < 3\}$.

(c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{1 < X < 2\}$.

(d) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{1 < X \le t | 1 < X < 2\}$ for $1 < t < 2$.

6. $\boxed{\text{60 points}}$ Suppose that X is a continuous random variable with density

$$f_X(t) = \begin{cases} \frac{1}{3} & \text{if } 0 < t < 1 \\ \frac{2}{3} & \text{if } 4 < t < 5 \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute the cumulative distribution function F_X of X.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$

(c) $\boxed{\text{10 points}}$ Compute $\text{Var}(X)$

(d) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[\max\{X, 3\}]$.

Suppose now that we define $Y \overset{\text{def}}{=} \max\{X, 3\}$.

(e) $\boxed{\text{5 points}}$ Graph the function $g(z) \overset{\text{def}}{=} \max\{z, 3\}$.

(f) $\boxed{\text{10 points}}$ Compute cumulative distribution F_Y of Y.

(g) $\boxed{\text{5 points}}$ Is Y a continuous random variable? Why or why not?

7. $\boxed{\text{40 points}}$ Suppose that X and Y are independent continuous random variables with densities

$$f_X(s) = \begin{cases} \frac{s^2}{2} e^{-s} & \text{if } s \ge 0 \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad f_Y(t) = \begin{cases} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

We first want to compute $\mathbb{P}\{X \le Y\}$

(a) $\boxed{\text{10 points}}$ Graph the region $s \le t$ on the (s,t)-plane

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X \le Y\}$

Define $Z \stackrel{\text{def}}{=} X + Y$. Let f_Z be the density of Z

(c) $\boxed{\text{10 points}}$ Compute $f_Z(4)$.

(d) $\boxed{\text{10 points}}$ Compute f_Z.

Answers

1. (a) $\binom{30}{7}$.
   (b) $\binom{29}{6}/\binom{30}{7}$.
   (c) $\binom{27}{3}/\binom{30}{7}$.
   (d)
$$\begin{aligned}&\mathbb{P}\{\text{All three brothers on committee}|\text{Alex on committee}\}\\ &\qquad = \frac{\mathbb{P}\{\text{all three brothers on committee and Alex on committee}\}}{\mathbb{P}\{\text{Alex on committee}\}} = \frac{\binom{27}{3}}{\binom{29}{6}}.\end{aligned}$$

2. Let
$$E = \{\text{woman has an ectopic pregnancy}\} \qquad \text{and} \qquad S = \{\text{woman is a smoker}\}.$$
Then $\mathbb{P}(E|S^c) = p$, $\mathbb{P}(E|S) = 2p$ and $\mathbb{P}(S) = .3$. We have that
$$\mathbb{P}(S|E) = \frac{\mathbb{P}(E|S)\mathbb{P}(S)}{\mathbb{P}(E|S)\mathbb{P}(S) + \mathbb{P}(E|S^c)\mathbb{P}(S^c)} = \frac{.3 \times 2p}{.3 \times 2p + .7p} = \frac{.6}{.6 + .7} = \frac{6}{13}.$$

3. (a) $2 \times 5! \times 4!/6! = \frac{1}{3}$.
   (b) $3 \times 2 \times 4!/6! = \frac{6}{5\times 6} = \frac{1}{5}$.
   (c) $4 \times 3! \times 3!/6! = \frac{1}{5}$.

4. (a) $1 - (35/36)^5$.
   (b) $1 - (35/36)^N$.
   (c) We want $1 - (35/36)^N \geq .9$.
$$N > \frac{\ln 10}{\ln(36/35)} > 82.$$

5. (a) $F_X(2) = \frac{11}{12}$.
   (b) $\mathbb{P}\{X < 3\} = F_X(3-) = \frac{11}{12}$.
   (c) $\mathbb{P}\{1 < X < 2\} = \mathbb{P}\{X < 2\} - \mathbb{P}\{X \leq 1\} = F_X(2-) - F_X(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.
   (d) For $1 < t < 2$,
$$\mathbb{P}\{1 < X \leq t\} = \mathbb{P}\{X \leq t\} - \mathbb{P}\{X \leq 1\} = F_X(t) - F_X(1) = \frac{1}{2} + \frac{t-1}{4} - \frac{1}{2} = \frac{t-1}{4}.$$
   Thus
$$\mathbb{P}\{1 < X \leq t|1 < X < 2\} = \frac{\mathbb{P}\{1 < X \leq t\}}{\mathbb{P}\{1 < X < 2\}} = \frac{\frac{1}{4}(t-1)}{\frac{1}{4}} = t - 1$$

6. (a)

$$F_X(t) = \int_{s=-\infty}^{t} f_X(s)ds = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{3}t & \text{if } 0 \le t < 1 \\ \frac{1}{3} & \text{if } 1 \le t < 4 \\ \frac{1}{3} + \frac{2}{3}(t-4) & \text{if } 4 \le t < 5 \\ 1 & \text{if } t \ge 4 \end{cases}$$

(b)

$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} t f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} tdt + \frac{2}{3}\int_{t=4}^{5} tdt = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{25-16}{2} = \frac{19}{6}.$$

(c)

$$\mathbb{E}[X^2] = \int_{t=-\infty}^{\infty} t^2 f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} t^2dt + \frac{2}{3}\int_{t=4}^{5} t^2dt$$
$$= \frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{125-64}{3} = \frac{123}{9} = \frac{41}{3};$$

thus $\mathrm{Var}(X) = \frac{123}{9} - \left(\frac{19}{6}\right)^2$.

(d)

$$\mathbb{E}[\max\{X,3\}] = \int_{t=-\infty}^{\infty} \max\{t,3\} f_X(t)dt = \frac{1}{3}\int_{t=0}^{1} 3dt + \frac{2}{3}\int_{t=4}^{5} tdt$$
$$= \frac{1}{3} \times 3 + \frac{2}{3} \times \frac{25-16}{2} = \frac{24}{6} = 4.$$

(e) $g(z) = 3$ if $z \le 3$, and $g(z) = z$ for $z \ge 3$.

(f) We have that

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 3 \\ F_X(t) & \text{if } t \ge 3 \end{cases} = \begin{cases} 0 & \text{if } t < 3 \\ \frac{1}{3} & \text{if } 3 \le t < 4 \\ \frac{1}{3} + \frac{2}{3}(t-4) & \text{if } 4 \le t < 5 \\ 1 & \text{if } t \ge 5 \end{cases}$$

(g) No; F_Y has a jump at 3.

7. (a) Above the line $s = t$.

(b)

$$\mathbb{P}\{X \le Y\} = \int_{s=-\infty}^{\infty}\int_{t=s}^{\infty} f_{X,Y}(s,t)dt\,ds = \int_{s=0}^{\infty}\int_{t=s}^{\infty} \frac{s^2}{2} e^{-s}e^{-t} ds\,dt$$
$$= \int_{s=0}^{\infty} \frac{s^2}{2} e^{-2s} ds$$
$$= \frac{1}{16}\int_{u=0}^{\infty} u^2 e^{-u} du = \frac{1}{16}\left\{ -u^2 e^{-u}\Big|_{u=0}^{\infty} + 2\int_{u=0}^{\infty} u e^{-u} du \right\} = \frac{1}{8}.$$

(c)

$$f_Z(4) = \int_{s=-\infty}^{\infty} f_X(s) f_Y(4-s) ds = \int_{s=0}^{4} \frac{s^2}{2} e^{-s} e^{-(4-s)} ds = e^{-4} \int_{s=0}^{4} \frac{s^2}{2} ds = \frac{4^3}{6} e^{-4}.$$

(d) $f_Z(t) = 0$ if $t < 0$. For $t > 0$,

$$f_Z(t) = \int_{s=-\infty}^{\infty} f_X(s) f_Y(t-s) ds = \int_{s=0}^{t} \frac{s^2}{2} e^{-s} e^{-(t-s)} ds = e^{-t} \int_{s=0}^{t} \frac{s^2}{2} ds = \frac{t^3}{6} e^{-t}.$$

Thus

$$f_Z(t) = \begin{cases} \frac{t^3}{6} e^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$