Math 461, Section X13 and X14, Fall 2009
Quiz 1, August 28 **Name:** ____________________

1. [10 points] Suppose that we want to arrange 15 textbooks on a shelf. There are

   - 6 math books
   - 5 history books
   - 4 chemistry books.

   (a) [5 points] How many ways can we arrange the textbooks if we want to keep each subject together?

   (b) [5 points] How many ways can we arrange the textbooks if we want a math book at each end?

Answers

1. (a) $6!5!4!3!$

   (b) $(6)_2 13!$

Math 461, Section X13 and X14, Fall 2009
Quiz 2, September 9 **Name:** ____________________

1. [10 points] This is problem 36 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards.
   (a) [5 points] What is the probability that both are aces?
   (b) [5 points] What is the probability that both have the same face value?

Answers

1. Set $p \stackrel{\text{def}}{=} 1/\binom{52}{2}$.
   (a) $\binom{4}{2}p$.
   (b) $13\binom{4}{2}p$.

Math 461, Section X13 and X14, Fall 2009
Quiz 3, September 11 **Name:** ____________________

1. [10 points] This is problem 18 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards. What is the probability that one of the cards is an ace and the other card is either a 10, jack, queen or king (i.e., what is the probability of a blackjack)?

Answers

1.

$$4 \times 16 / \binom{52}{2}.$$

Math 461, Section X13 and X14, Fall 2009
Quiz 4, September 16 **Name:** ____________________

1. [10 points] This is essentially problem 35 from the book. It is your birthday! Time to do some math. Mom or Dad have hidden a present for you, either upstairs or downstairs. With probability .6, the present was hidden by Mom. If Nom has hidden the present, it is upstairs with probability .7. Dad is equally likely to have hidden it upstairs or downstairs.
   (a) [5 points] What is the probability that the present is upstairs?
   (b) [5 points] If it is upstairs, what is the probability that it was hidden by Dad?

Answers

1.
$$M = \{\text{Mom hid the present}\} \qquad \text{and} \qquad U = \{\text{present is upstairs}\}.$$

We know that

$$\mathbb{P}(M) = .6, \qquad \mathbb{P}(U|M) = .7, \qquad \text{and} \qquad \mathbb{P}(U|D) = .5.$$

We then compute that

$$\mathbb{P}(U \cap M) = \mathbb{P}(U|M)\mathbb{P}(M) = .7 \times .6 = .42$$
$$\mathbb{P}(U \cap D) = \mathbb{P}(U|D)\mathbb{P}(D) = .5 \times .4 = .20$$

(a) $\mathbb{P}(U) = \mathbb{P}(U \cap M) + \mathbb{P}(U \cap D) = .42 + .2 = .62.$

(b) $\mathbb{P}(D|U) = \mathbb{P}(U \cap D)/\mathbb{P}(U) = .2/.62.$

Math 461, Section X13 and X14, Fall 2009
Quiz 5, September 23 **Name:** ____________________

1. [10 points] Suppose we have an unfair coin with $\mathbb{P}\{H\} = p$ and $\mathbb{P}\{T\} = 1 - p$. We flip the coin a countable sequence of times. We are interested in the statistics of the third heads.
   (a) [2 points] Pick a specific configuration where the third heads is on the 9th toss.
   (b) [2 points] What is the probability of the configuration chosen in part a)?
   (c) [3 points] What is the probability that the third heads is on the 9th toss?
   (d) [2 points] For any integer $n \geq 3$, what is the probability that the third heads is on the n-th toss.

Answers

1. (a) $(TTTHTHTTH)$

   (b) $p^3(1-p)^6$.

   (c) $\binom{8}{2}p^3(1-p)^6$.

   (d) $\binom{n-1}{2}p^3(1-p)^{n-3}$.

Math 461, Section X13 and X14, Fall 2009
Quiz 6, September 30 **Name:** ____________________

1. [10 points] A box has balls labelled B_1 through B_{20}. Pick three balls. Let X be the number of the middle ball. In other words, if we pick balls B_5, B_3 and B_{17}, we order them and the middle ball is B_5, so $X = 5$.

   (a) [2 points] Verbally describe what it means that $X = 5$.

   (b) [2 points] Find $\mathbb{P}\{X = 5\}$.

   (c) [2 points] Find $\mathbb{P}\{X = 9\}$.

   (d) [2 points] Find the possible values of X.

   (e) [2 points] Find the probability mass function of X; i.e., find $\mathbb{P}\{X = n\}$.

Answers

1. (a) There is one ball among $\{B_1, B_2 \dots B_4\}$, we picked B_5, and there is one ball among $\{B_5 \dots B_{20}\}$.

   (b) $4 \times (20-5)/\binom{20}{3}$.

   (c) $8 \times (20-9)/\binom{20}{3}$.

   (d) $\{2, 3 \dots 19\}$.

   (e)

$$p_X(n) = \begin{cases} \frac{(n-1)(20-n)}{\binom{20}{3}} & \text{if } n \in \{2, 3 \dots 19\} \\ 0 & \text{else} \end{cases}$$

Math 461, Section X13 and X14, Fall 2009
Quiz 7, October 7 **Name:** ______________

1. [10 points] Let X be geometric with parameter p; i.e.,

$$p_X(n) = \begin{cases} (1-p)^{n-1}p & \text{if } n \in \{1, 2 \dots\} \\ 0 & \text{else} \end{cases}$$

Define the function $f(x) \overset{\text{def}}{=} \max\{x, 10\}$ for all $x \in \mathbb{R}$. Define $Y \overset{\text{def}}{=} f(X)$.

(a) [2 points] Graph f.

(b) [2 points] Compute $\mathbb{P}\{X = 5\}$.

(c) [2 points] Compute $\mathbb{P}\{X = 15\}$.

(d) [2 points] Compute $\mathbb{P}\{X = 10\}$.

(e) [2 points] Compute the probability mass function of Y.

Answers

1. (a) Flat line at height 10, then up with slope 1.

   (b) $\mathbb{P}\{Y=5\}=0$.

   (c) $\mathbb{P}\{Y=15\}=\mathbb{P}\{X=15\}=(1-p)^{14}p$.

   (d)

$$\mathbb{P}\{Y=10\}=\mathbb{P}\{X\le 10\}=1-\mathbb{P}\{X\ge 11\}=1-\sum_{n=11}^{\infty}(1-p)^{n-1}p=1-(1-p)^{10}.$$

   (e)

$$p_Y(n)=\begin{cases}1-(1-p)^{10} & \text{if } n=10\\ (1-p)^{n-1}p & \text{if } n\in\{11,12\dots\}\\ 0 & \text{else}\end{cases}$$

Math 461, Section X13 and X14, Fall 2009

Quiz 8, October 19 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Let X be a random variable with probability mass function

$$p_X(n) \overset{\text{def}}{=} \begin{cases} \frac{1}{12} & \text{if } n = 2 \\ \frac{2}{12} & \text{if } n = 1 \\ \frac{3}{12} & \text{if } n = 0 \\ \frac{5}{12} & \text{if } n = -1 \\ \frac{1}{12} & \text{if } n = -2 \end{cases}$$

   (a) $\boxed{\text{2 points}}$ Compute $\mathbb{E}[X]$

   (b) $\boxed{\text{2 points}}$ Compute $\mathbb{E}[X^2]$

   (c) $\boxed{\text{3 points}}$ Compute the variance of X.

   (d) $\boxed{\text{3 points}}$ Compute $\mathbb{E}\left[e^{5X}\right]$. I want to see that you know what this means; I know that you can't explicitly evaluate exponentials without a calculator.

Answers

1. (a)

$$\mathbb{E}[X] = \frac{2\times 1 + 1\times 2 + 0\times 3 + (-1)\times 5 + (-2)\times 1}{12} = \frac{-3}{12}.$$

(b)

$$\mathbb{E}[X^2] = \frac{2^2\times 1 + 1^2\times 2 + 0^2\times 3 + (-1)^2\times 5 + (-2)^2\times 1}{12} = \frac{15}{12}.$$

(c)

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{15}{12} - \left(\frac{-3}{12}\right)^2 = \frac{15\times 12 - 9}{144} = \frac{180-9}{144} = \frac{171}{144}.$$

(d)

$$\begin{aligned}\mathbb{E}\left[e^{5X}\right] &= \frac{e^{10}\times 1 + e^{5}\times 2 + 1\times 3 + e^{-5}\times 5 + e^{-10}\times 1}{12}\\ &= \frac{e^{10} + 2e^{5} + 3 + 5e^{-5} + e^{-10}}{12}.\end{aligned}$$

Math 461, Section X13 and X14, Fall 2009
Quiz 9, November 4 **Name:** ____________________

1. 10 points Let X be a continuous random variable with density

$$f_X(t) \overset{\text{def}}{=} \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Compute the moment generating function $\Phi(\theta) \overset{\text{def}}{=} \mathbb{E}[e^{\theta X}]$ for $\theta < 5$.

Answers

1. For $\theta < 5$,

$$\Phi(\theta) = 5 \int_{t=0}^{\infty} \exp\left[-(5-\theta)t\right] dt = \frac{5}{5-\theta}$$

Math 461, Section X13 and X14, Fall 2009
Quiz 10, November 6 Name: ____________

1. [10 points] Let U be a Uniform$(0,1)$ continuous random variable; i.e., it has density

$$f_U(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$g(u) \overset{\text{def}}{=} \begin{cases} 1-u & \text{if } u \le \frac{2}{3} \\ u & \text{if } u > \frac{2}{3} \end{cases}$$

and define $X \overset{\text{def}}{=} g(U)$.

(a) [5 points] Compute F_X, the cumulative distribution function of X.

(b) [5 points] Compute f_X, the density of X.

Answers

1. (a) $F_X(t) = 1$ if $t \ge 1$ and $F_X(t) = 0$ for $t < 0$. For $\frac{1}{3} \le t < \frac{2}{3}$, we have that

$$F_X(t) = \mathbb{P}\left\{1 - t \le U \le \frac{2}{3}\right\} = \frac{2}{3} - (1-t) = t - \frac{1}{3}.$$

For $\frac{2}{3} \le t < 1$, we have that

$$F_X(t) = \mathbb{P}\{1 - t \le U \le t\} = t - (1-t) = 2t - 1.$$

Thus

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ t - \frac{1}{3} & \text{if } \frac{1}{3} \le t < \frac{2}{3} \\ 2t - 1 & \text{if } \frac{2}{3} \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(b) Differentiate F_X to get that

$$f_X(t) = \begin{cases} 1 & \text{if } \frac{1}{3} < t < \frac{2}{3} \\ 2 & \text{if } \frac{2}{3} < t < 1 \\ 0 & \text{else} \end{cases}$$

Math 461, Section X13 and X14, Fall 2009
Quiz 11, November 11 **Name:** ____________

1. [10 points] X be an $N(0,1)$ Gaussian; i.e., a continuous random variable with density

$$f_X(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \qquad t \in \mathbb{R}$$

Set $Y \overset{\text{def}}{=} X + 7$.

   (a) [5 points] Compute $\mathbb{P}\{Y \le 13\}$ in terms of an integral of f_X.

   (b) [5 points] Compute the cumulative distribution function of Y in terms of an integral of f_X.

Answers

1. (a)

$$\mathbb{P}\{Y \leq 13\} = \mathbb{P}\{X + 7 \leq 13\} = \mathbb{P}\{X \leq 6\} = \int_{s=-\infty}^{6} f_X(s) ds$$

(b) For $t \in \mathbb{R}$

$$F_Y(t) = \mathbb{P}\{X + 7 \leq t\} = \mathbb{P}\{X \leq t - 7\} = \int_{s=-\infty}^{t-7} f_X(s) ds.$$

Math 461, Section X13 and X14, Fall 2009
Quiz 12, November 20 **Name:** ____________________

1. [10 points] Evaluate the integral

$$\int_{s=0}^{5} 0 ds.$$

Answers

1. 0.

Math 461, Section X13 and X14, Fall 2009
Exam 1, September 25

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. [45 points] This is an extension of Wednesday's quiz. Suppose we have an unfair coin with $\mathbb{P}\{H\} = p$ and $\mathbb{P}\{T\} = 1 - p$. We flip the coin a countable sequence of times. We are interested in the statistics of the first and third heads.
   (a) [5 points] Pick a specific configuration where the third heads is on the 9th toss.
   (b) [5 points] What is the probability of the configuration chosen in part a)?
   (c) [5 points] What is the probability that the third heads is on the 9th toss?
   (d) [5 points] For any integer $n \geq 3$, what is the probability that the third heads is on the n-th toss.
   (e) [5 points] Pick a specific configuration where the first heads is on the 4th toss and the third heads is on the 9th toss.
   (f) [5 points] What is the probability of the configuration chosen in part e)?
   (g) [5 points] What is the probability that the first heads is on the 4th toss and the third heads is on the 9th toss.
   (h) [5 points] Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the 4th toss?
   (i) [5 points] Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the n-th toss, for each $n \in \{1, 2 \dots 7\}$.

2. [20 points] Suppose that

$$\mathbb{P}(A) = .3 \qquad \mathbb{P}(B) = .6 \qquad \mathbb{P}\left((A \cup B)^c\right) = .28.$$

   (a) [5 points] Compute $\mathbb{P}(A \cup B)$.
   (b) [5 points] Compute $\mathbb{P}(A \cap B)$.
   (c) [5 points] Compute $\mathbb{P}(A \setminus B)$.
   (d) [5 points] Are A and B independent?

3. [21 points] Suppose that Jack, Betty, and Rick are in a team of 30 people. Two committees will be formed; committee A contains 7 people and committee B contains of 9 people. Assume that all committees are equally likely. Assume also that the committees can overlap.
   (a) [7 points] Find the probability that Jack and Rick are both on committee A.
   (b) [7 points] Find the probability that Jack and Rick are both on committee A and Jack and Betty are on the same committee.

(c) 7 points Find the probability that Jack and Betty are both on the same committee given that Jack and Rick are on committee A.

4. 14 points This question is motivated by question 3.19 in chapter 3. A "quit smoking" class is 40% male. A total of 30% of the women who took the class remained nonsmokers for a year. Similarly, a total of 20% of men who took the class remained nonsmokers for a year. A party was held at the end of the year for those who had quit for the year.

   (a) 7 points What percentage of the original class was at the party?

   (b) 7 points What percent of those attending the party were male?

1. (a) $(HTTTTTHTH)$
   (b) $(1-p)^6p^3$
   (c) $\binom{8}{2}(1-p)^6p^3$.
   (d) $\binom{n-1}{2}(1-p)^{n-3}p^3$.
   (e) $(TTTHTHTTH)$
   (f) $(1-p)^6p^3$.
   (g) $4(1-p)^6p^3$.
   (h) $\frac{4}{\binom{8}{2}}$.
   (i) $\frac{8-n}{\binom{8}{2}}$.
2. (a) $\mathbb{P}(A\cup B) = 1-\mathbb{P}\left((A\cup B)^c\right) = 1-.28 = .72$.
   (b) $\mathbb{P}(A\cap B) = \mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cup B) = .3+.6-.72 = .9-.72 = .18$.
   (c) $\mathbb{P}(A\setminus B) = \mathbb{P}(A)-\mathbb{P}(A\cap B) = .3-.18 = .12$.
   (d) Yes; $\mathbb{P}(A\cap B) = .18 = .3\times .6 = \mathbb{P}(A)\mathbb{P}(B)$.
3.

$$C = \{\text{Jack and Rick are on committee A}\}$$
$$D = \{\text{Jack and Betty are on the same committee}\}.$$

(a)

$$\mathbb{P}(C) = \frac{\binom{28}{5}}{\binom{30}{7}} = \frac{\binom{28}{5}\binom{30}{9}}{\binom{30}{7}\binom{30}{9}}$$

(b) The challenge here is to write $C\cap D$ as a disjoint union. Let's partition the space based on whether or not Betty is on committee A. We have $C\cap D = E_1\cup E_2$ where

$$E_1 = C\cap D\cap\{\text{Betty is on A}\}$$
$$E_2 = C\cap D\cap\{\text{Betty is not on A}\}.$$

Note that

$$C\cap\{\text{Betty is on A}\}\subset D$$

so $E_1 = \{\text{Jack, Rick and Betty are on A}\}$. Secondly, if Betty is not on commmittee A but she and Jack are on the same committee, then they must both be on B. In other words,

$$E_2 = \{\text{Jack and Rick are on committee A, but Betty is not and Jack and Betty are on committee B}\}.$$

Then

$$\mathbb{P}(E_1) = \frac{\binom{27}{4}}{\binom{30}{7}}$$
$$\mathbb{P}(E_2) = \frac{\binom{27}{5}\binom{28}{7}}{\binom{30}{7}\binom{30}{9}}$$

so

$$\mathbb{P}(C \cap D) = \frac{\binom{27}{4}}{\binom{30}{7}} + \frac{\binom{27}{5}\binom{28}{7}}{\binom{30}{7}\binom{30}{9}}.$$

Alternately, $\mathbb{P}(C \cap D) = \mathbb{P}(C) - \mathbb{P}(C \setminus D)$ and

$$C \setminus D = \{\text{Jack and Rick are on committee A and Betty is not on committee A}\}.$$

Here

$$\mathbb{P}(C \cap D) = \frac{\binom{28}{5}}{\binom{30}{7}} - \frac{\binom{27}{5}}{\binom{30}{7}}$$

(c)

$$\mathbb{P}(D|C) = \frac{\binom{27}{4}\binom{30}{9} + \binom{27}{5}\binom{28}{7}}{\binom{28}{5}\binom{30}{9}}$$

4.

$$M = \{\text{men who were in the class}\} \qquad Q = \{\text{quit smoking for a year}\}.$$

We have

$$\mathbb{P}(M) = .4 \qquad \mathbb{P}(Q|M^c) = .3 \qquad \mathbb{P}(Q|M) = .2.$$

We have that

$$\mathbb{P}(Q \cap M) = \mathbb{P}(Q|M)\mathbb{P}(M) = .2 \times .4 = .08$$
$$\mathbb{P}(Q \cap M^c) = \mathbb{P}(Q|M^c)\mathbb{P}(M^c) = .3 \times .6 = .18.$$

(a)

$$\mathbb{P}(Q) = \mathbb{P}(Q \cap M) + \mathbb{P}(Q \cap M^c) = .08 + .18 = .26.$$

(b)

$$\mathbb{P}(M|Q) = \frac{\mathbb{P}(Q \cap M)}{\mathbb{P}(Q)} = \frac{.08}{.26}.$$

Math 461, Section X13 and X14, Fall 2009
Exam 2, October 26

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. [28 points] Let X be a random variable with probability mass function

$$p_X(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{15} & \text{if } n = 3 \\ \frac{2}{15} & \text{if } n = 2 \\ \frac{2}{15} & \text{if } n = 1 \\ \frac{3}{15} & \text{if } n = 0 \\ \frac{5}{15} & \text{if } n = -1 \\ \frac{2}{15} & \text{if } n = -2 \end{cases}$$

Define $Y = X^2$.

   (a) [6 points] Compute $\mathbb{E}[X]$

   (b) [6 points] Compute $\mathbb{E}[X^2]$

   (c) [8 points] Compute $\mathbb{P}\{Y = 4\}$

   (d) [8 points] Compute p_Y, the probability mass function of Y.

2. [12 points] Assume that we have a coin with bias $\mathbb{P}\{H\} = \frac{2}{10}$ (i.e., the probability of getting a heads is $\frac{2}{10}$). Flip the coin 10 times. Define the random variable

$$X = \begin{cases} 1 & \text{if the first, second, and third tosses come up heads} \\ 0 & \text{else} \end{cases}$$

   (a) [4 points] Compute $\mathbb{E}[X]$

   (b) [3 points] Compute $\mathbb{E}[X^2]$

   (c) [2 points] Compute $\mathbb{E}[X^3]$

   (d) [3 points] Compute the variance of X.

3. [19 points] Take a coin, and define $p \stackrel{\text{def}}{=} \mathbb{P}\{\text{heads}\}$. Flip the coin until the third heads appears; let X be the position of the third heads.

   (a) [4 points] Write down a specific configuration such that $X = 7$.

   (b) [4 points] Compute the probability of the configuration in part a.

   (c) [5 points] Compute the probability that $X = 7$.

   (d) [6 points] Compute the probability mass function for X.

4. $\boxed{\text{17 points}}$ Suppose that $\mathbb{E}[X]=2$ and the variance of X is 5. Define $Y=7X+1$

   (a) $\boxed{\text{5 points}}$ Compute the standard deviation of X.

   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X^2]$.

   (c) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[Y]$.

   (d) $\boxed{\text{2 points}}$ Compute $\mathbb{E}[Y^2]$.

5. $\boxed{\text{21 points}}$ Let X be a random variable with moment generating function

$$\Phi(\theta) \stackrel{\text{def}}{=} \mathbb{E}\left[e^{\theta X}\right] = \begin{cases} \frac{5}{5-\theta} & \text{if } \theta < 5 \\ \infty & \text{else} \end{cases}$$

   (a) $\boxed{\text{6 points}}$ Compute $\mathbb{E}[X]$.

   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X^2]$.

   (c) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X^3]$.

   (d) $\boxed{\text{5 points}}$ Compute the variance of X.

6. $\boxed{\text{3 points}}$ Suppose that Timmy Typewriter is typesetting a book. On any given letter, he typically makes an error with probability .00007. We assume that the letters are independent, there are 400 letters on a page and 10 pages in a chapter. Use the Poisson approximation to approximate the probability that there will be at most 3 errors in the first chapter.

Answers

1. (a) $\mathbb{E}[X] = \frac{3\times1+2\times2+1\times2+0\times3+(-1)\times5+(-2)\times2}{15} = 0.$
   (b) $\mathbb{E}[X^2] = \frac{3^2\times1+2^2\times2+1^2\times2+0^2\times3+(-1)^2\times5+(-2)^2\times2}{15} = \frac{32}{15}.$
   (c) $\mathbb{P}\{Y=4\} = \mathbb{P}\{X=2\} + \mathbb{P}\{X=-2\} = \frac{4}{15}.$
   (d)
$$p_Y(n) = \begin{cases} \frac{1}{15} & \text{if } n=9 \\ \frac{4}{15} & \text{if } n=4 \\ \frac{7}{15} & \text{if } n=1 \\ \frac{3}{15} & \text{if } n=0 \end{cases}$$

2. Note that $X = X^2 = X^3$.
   (a) $\mathbb{E}[X] = 1 \times \left(\frac{2}{10}\right)^3 + 0 \times \left(1 - \left(\frac{2}{10}\right)^3\right) = \frac{8}{1000}$
   (b) $\mathbb{E}[X^2] = \frac{8}{1000}$
   (c) $\mathbb{E}[X^3] = \frac{8}{1000}$
   (d) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{8}{1000} - \left(\frac{8}{1000}\right)^2 = \frac{8000-64}{10^6} = \frac{7936}{10^6}.$

3. (a) $(TTHHTTH)$
   (b) $p^3(1-p)^4.$
   (c) $\binom{6}{2}p^3(1-p)^4.$
   (d)
$$p_X(n) = \begin{cases} \binom{n-1}{2}p^3(1-p)^{n-3} & \text{if } n \in \{3,4\dots\} \\ 0 & \text{else} \end{cases}$$

4. (a) $\sqrt{5}.$
   (b) $\mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 5+4 = 9.$
   (c) $\mathbb{E}[Y] = 7\mathbb{E}[X] + 1 = 7\times2+1 = 15.$
   (d) $\mathbb{E}[Y^2] = \mathbb{E}[49X^2 + 14X + 1] = 49\mathbb{E}[X^2] + 14\mathbb{E}[X] + 1 = 49\times9 + 14\times2 + 1 = 470.$

5. For $\theta < 5$,
$$\Phi'(\theta) = \frac{5}{(5-\theta)^2} \qquad \Phi^{(2)}(\theta) = \frac{2\times5}{(5-\theta)^3} \qquad \Phi^{(2)}(\theta) = \frac{3!\times5}{(5-\theta)^3}$$
   (a) $\mathbb{E}[X] = \Phi'(0) = \frac{1}{5}.$
   (b) $\mathbb{E}[X^2] = \Phi^{(2)}(0) = \frac{2}{25}.$
   (c) $\mathbb{E}[X^3] = \Phi^{(3)}(0) = \frac{6}{125}.$
   (d) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{25} - \frac{1}{25} = \frac{1}{25}.$

6. We assume that the errors are Poisson distributed with parameter $\lambda = .00007 \times 4000 = 2.8$. Thus

$$\mathbb{P}\{X \le 3\} = \sum_{n=0}^{3} e^{-2.8}\frac{(2.8)^n}{n!} = e^{-2.8}\left\{1 + 2.8 + \frac{(2.8)^2}{2} + \frac{(2.8)^3}{6}\right\}.$$

Math 461, Section X13 and X14, Fall 2009
Exam 3, November 18

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

For the purpose of this exam, we will define

$$\Phi(t) \stackrel{\text{def}}{=} \int_{s=-\infty}^{t} \frac{\exp\left[-\frac{s^2}{2}\right]}{\sqrt{2\pi}} ds \qquad t \in \mathbb{R}$$

1. 15 points Let X be a random variable with cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{5} & \text{if } 0 \le t < 1 \\ \frac{1}{2} + \frac{t-1}{5} & \text{if } 1 \le t < 2 \\ \frac{5}{6} & \text{if } 2 \le t < 3 \\ 1 & \text{if } t \ge 3 \end{cases}$$

   (a) 5 points Compute $\mathbb{P}\{X = 1\}$.

   (b) 5 points Compute $\mathbb{P}\{X = 2\}$

   (c) 5 points Compute $\mathbb{P}\{X < 3\}$.

2. 31 points Let X be a continuous random variable with density function

$$f_X(t) = \begin{cases} e^{-t} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   (a) 5 points Compute $\mathbb{P}\{X \ge 7\}$

   (b) 5 points Compute $\mathbb{P}\{X \ge 3\}$

   (c) 5 points Compute $\mathbb{P}\{X \ge 7 | X \ge 3\}$

   Define $Y \stackrel{\text{def}}{=} X^2$

   (d) 5 points Compute $\mathbb{P}\{Y \le 4\}$

   (e) 5 points Compute the cumulative distribution function of Y.

   (f) 6 points Is Y continuous? If so, compute its density. If not, state why it is not continuous.

3. 54 points Let X be a Gaussian with mean 1 and variance 4. Let N be a Gaussian with mean 0 and variance 1.

(a) $\boxed{\text{5 points}}$ Write X in terms of N.

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \geq 3\}$ using the attached table.

(c) $\boxed{\text{6 points}}$ Compute $\mathbb{P}\{|X| \geq 3\}$ using the attached table.

(d) $\boxed{\text{5 points}}$ What is the density of X?

Define a function

$$\varphi(z) \stackrel{\text{def}}{=} \begin{cases} -1 & \text{if } z \leq 0 \\ 2z & \text{if } 0 < z < 3 \\ 6 & \text{if } z \geq 3 \end{cases}$$

(e) $\boxed{\text{5 points}}$ Graph φ.

Define $Y \stackrel{\text{def}}{=} \varphi(X)$. As necessary, answer the following questions in terms of Φ, not the table.

(f) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{Y \leq -2\}$

(g) $\boxed{\text{6 points}}$ Compute $\mathbb{P}\{Y \leq 3\}$

(h) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{Y \leq 7\}$

(i) $\boxed{\text{7 points}}$ What is the cumulative distribution function of Y (in terms of Φ as needed).

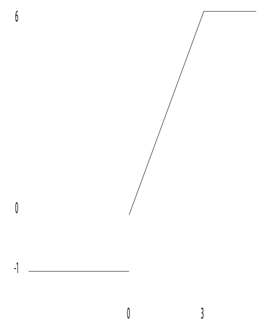
(j) $\boxed{\text{5 points}}$ Is Y a continuous random variable? If not, why?

Answers

1. (a)
$$\mathbb{P}\{X=1\} = F_X(1) - F_X(1-) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$
(b)
$$\mathbb{P}\{X=2\} = F_X(2) - F_X(2-) = \frac{5}{6} - \frac{7}{10} = \frac{4}{30} = \frac{2}{15}.$$
(c) $\mathbb{P}\{X<3\} = F_X(3-) = \frac{5}{6}$.

2. (a) $\mathbb{P}\{X \ge 7\} = \int_{t=7}^{\infty} f_X(t)dt = \int_{t=7}^{\infty} e^{-t}dt = e^{-7}$.
(b) $\mathbb{P}\{X \ge 3\} = \int_{t=3}^{\infty} f_X(t)dt = \int_{t=3}^{\infty} e^{-t}dt = e^{-3}$.
(c)
$$\mathbb{P}\{X \ge 7 | X \ge 3\} = \frac{\mathbb{P}\{X \ge 7,\, X \ge 3\}}{\mathbb{P}\{X \ge 3\}} = \frac{\mathbb{P}\{X \ge 7\}}{\mathbb{P}\{X \ge 3\}} = \frac{e^{-7}}{e^{-3}} = e^{-4}.$$
(d) $\mathbb{P}\{Y \le 4\} = \mathbb{P}\{|X| \le 2\} = \int_{t=0}^{2} e^{-t}dt = 1 - e^{-2}$.
(e)
$$F_Y(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 - e^{-\sqrt{t}} & \text{if } t \ge 0 \end{cases}$$
(f) Y is continuous and
$$f_Y(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{e^{-\sqrt{t}}}{2\sqrt{t}} & \text{if } t>0 \end{cases}$$

3. (a) $X = 2N+1$.
(b)
$$\begin{aligned}\mathbb{P}\{X \ge 3\} &= \mathbb{P}\{2N+1 \ge 3\} = \mathbb{P}\{N \ge 1\} \\ &= 1 - \mathbb{P}\{N<1\} = 1 - \Phi(1) = 1 - .8413 = .1587\end{aligned}$$
(c)
$$\begin{aligned}\mathbb{P}\{|X| \ge 3\} &= \mathbb{P}\{X \le -3\} + \mathbb{P}\{X \ge 3\} = \mathbb{P}\{2N+1 \le -3\} + \mathbb{P}\{2N+1 \ge 3\} \\ &= \mathbb{P}\{N \le -2\} + \mathbb{P}\{N \ge 1\} = \mathbb{P}\{N \ge 2\} + \mathbb{P}\{N \ge 1\} \\ &= 1 - \mathbb{P}\{N \le 2\} + 1 - \mathbb{P}\{N \le 1\} = 2 - \Phi(2) - \Phi(1) = 2 - .9772 - .8413 = .1815\end{aligned}$$
(d)
$$f_X(t) = \frac{\exp\left[-\frac{(t-1)^2}{8}\right]}{\sqrt{8\pi}} \qquad t \in \mathbb{R}$$
(e)



(f) $\mathbb{P}\{Y \leq -2\} = \mathbb{P}(\emptyset) = 0$

(g)
$$\mathbb{P}\{Y \leq 3\} = \mathbb{P}\left\{X \leq \frac{3}{2}\right\} = \mathbb{P}\left\{2N+1 \leq \frac{3}{2}\right\} = \mathbb{P}\left\{N \leq \frac{1}{4}\right\} = \Phi\left(\frac{1}{4}\right)$$

(h) $\mathbb{P}\{Y \leq 7\} = 1$.

(i)
$$F_Y(t) = \begin{cases} 0 & \text{if } t < -1 \\ \Phi\left(-\frac{1}{2}\right) & \text{if } -1 \leq t < 0 \\ \Phi\left(\frac{t/2-1}{2}\right) & \text{if } 0 \leq t < 6 \\ 1 & \text{if } t \geq 6 \end{cases}$$

Note that $\Phi\left(-\frac{1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right)$.

(j) No; it has a jump at -1 and 6.

Math 461, Section X13 and X14, Fall 2009
Final, December 15

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
200 points total.

1. [20 points] This is question 7 on page 16 of the book.
   (a) [5 points] How many ways can we seat 3 boys and 3 girls in a row?
   (b) [5 points] How many ways can we seat 3 boys and 3 girls in a row if the boys and girls are each to sit together?
   (c) [5 points] How many ways can we seat 3 boys and 3 girls in a row if only the boys must sit together?
   (d) [5 points] How many ways can we seat 3 boys and 3 girls in a row if no two people of the same gender are allowed to sit together?

2. [15 points] At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.
   (a) [10 points] Randomly select a computer science student. What is the probability that it is a woman?
   (b) [5 points] In this data, is gender independent of the decision to study computer science?

3. [12 points] Let X be a random variable with probability mass function

$$p(j) = \begin{cases} \frac{2}{9} & \text{if } j = -1 \\ \frac{1}{9} & \text{if } j = 0 \\ \frac{2}{9} & \text{if } j = 3 \\ \frac{4}{9} & \text{if } j = 7 \\ 0 & \text{else} \end{cases}$$

   (a) [6 points] Compute $\mathbb{E}[X]$.
   (b) [6 points] Compute $\mathbb{E}[\min\{X, 1\}]$.

4. [40 points] We will here develop some ideas relating to the *negative hypergeometric* distribution. This is similar in some ways to a negative binomial, and you might keep those calculations in mind.

   Suppose that we have a box containing 20 black balls $\{B_1, B_2 \dots B_{20}\}$ and 30 red balls $\{R_1, R_2 \dots R_{30}\}$. Let's pick balls out of the box, one by one.

   Let X_7 be the first time that we have 7 black balls.

(a) $\boxed{\text{10 points}}$ Write a specific configuration where $X_7 = 13$

(b) $\boxed{\text{10 points}}$ Compute the probability of this configuration.

(c) $\boxed{\text{10 points}}$ Compute the probability that $\mathbb{P}\{X_7 = 13\}$.

(d) $\boxed{\text{10 points}}$ What is the probability mass function of X_7?

5. $\boxed{\text{66 points}}$ Let U be a continuous random variable which is uniformly distributed on $(0, 1)$; i.e., it has density

$$f_U(t) = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Define $\varphi_1(u) \stackrel{\text{def}}{=} u^3$ for $u \in (0, 1)$, and define $X_1 \stackrel{\text{def}}{=} \varphi_1(U)$.

(a) $\boxed{\text{5 points}}$ Graph φ_1

(b) $\boxed{\text{7 points}}$ Compute $\mathbb{P}\left\{X_1 \le \frac{2}{3}\right\}$.

(c) $\boxed{\text{5 points}}$ Compute the cumulative of X_1.

(d) $\boxed{\text{4 points}}$ Does X_1 have a density? If so, compute it. If not, explain why.

Define $\varphi_2(u) \stackrel{\text{def}}{=} 1/u^3$ for $u \in (0, 1)$, and define $X_2 \stackrel{\text{def}}{=} \varphi_2(U)$.

(e) $\boxed{\text{5 points}}$ Graph φ_2

(f) $\boxed{\text{7 points}}$ Compute $\mathbb{P}\{X_2 \le 5\}$.

(g) $\boxed{\text{5 points}}$ Compute the cumulative of X_2.

(h) $\boxed{\text{4 points}}$ Does X_2 have a density? If so, compute it. If not, explain why.

Define $\varphi_3(u) \stackrel{\text{def}}{=} -\frac{1}{5}\ln u$ for $u \in (0, 1)$, and define $X_3 \stackrel{\text{def}}{=} \varphi_3(U)$.

(i) $\boxed{\text{5 points}}$ Graph φ_3

(j) $\boxed{\text{7 points}}$ Compute $\mathbb{P}\{X_3 \le 7\}$.

(k) $\boxed{\text{5 points}}$ Compute the cumulative of X_3.

(l) $\boxed{\text{4 points}}$ Does X_3 have a density? If so, compute it. If not, explain why.

(m) $\boxed{\text{3 points}}$ X_3 is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

6. $\boxed{\text{47 points}}$ Fix two parameters $p \in (0, 1)$ and $\lambda > 0$. Consider two discrete random variables X and Y with joint probability mass function

$$p_{X,Y}(i,j) = \begin{cases} \binom{j}{i} p^i (1-p)^{j-i} e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, 1 \dots\} \text{ and } i \in \{0, 1 \dots j\} \\ 0 & \text{else} \end{cases}$$

(as usual, we define $\binom{0}{0} \stackrel{\text{def}}{=} 1$). In other words,

$$\mathbb{P}\{X = 2, Y = 7\} = \binom{7}{2} p^2 (1-p)^5 e^{-\lambda} \frac{\lambda^7}{7!}.$$

This is some combination of binomial and Poisson distributions, of course.

(a) 10 points Compute $p_X(5) \stackrel{\text{def}}{=} \mathbb{P}\{X = 5\}$. Hint: recall that $\binom{j}{i} = \frac{j!}{i!(j-i)!}$. Recall also the Taylor formula for the exponential.

(b) 5 points Compute the probability mass function of X.

(c) 3 points X is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

(d) 10 points Compute $p_Y(7) \stackrel{\text{def}}{=} \mathbb{P}\{Y = 7\}$ Hint: recall the binomial theorem.

(e) 5 points Compute the probability mass function of Y.

(f) 3 points Y is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

(g) 4 points Are X and Y independent?

(h) 7 points Compute $\mathbb{P}\{X = 5 | Y = 7\}$.

Answers

1. (a) 6!
   (b) $2 \times 3! \times 3!$ (the 2 comes from which gender is left-most)
   (c) $4! \times 3!$ (group the boys as one unit, and then order within the unit)
   (d) $2 \times 3! \times 3!$ (the two comes from which gender is left-most).

2.

$$W = \{\text{woman}\} \qquad \text{and} \qquad C = \{\text{computer science}\}.$$

   (a)

$$\mathbb{P}(W|C) = \frac{\mathbb{P}(W \cap C)}{\mathbb{P}(C)} = \frac{7}{10} = .7.$$

   (b) No; $\mathbb{P}(W|C) = .7 \neq .54 = \mathbb{P}(W)$.

3. (a)

$$\mathbb{E}[X] = \frac{(-1) \times 2 + 0 \times 1 + 3 \times 2 + 7 \times 4}{9} = \frac{32}{9}.$$

   (b)

$$\begin{aligned}\mathbb{E}[\min\{X,1\}] &= \sum_j \min\{j,1\}p(j) \\ &= \min\{-1,1\}\frac{2}{9} + \min\{0,1\}\frac{1}{9} + \min\{3,1\}\frac{2}{9} + \min\{7,1\}\frac{4}{9} \\ &= (-1)\frac{2}{9} + (0)\frac{1}{9} + (1)\frac{2}{9} + (1)\frac{4}{9} = \frac{-2+0+2+4}{9} = \frac{4}{9}.\end{aligned}$$

4. (a) $(B_4, B_2, R_9, R_{11}, R_3, B_3, R_7, B_{20}, B_{15}, R_{30}, B_8, R_1, B_1)$
   (b)

$$\frac{1}{(50)_{13}}.$$

   (c)

$$\binom{12}{6}\frac{(20)_7(30)_6}{(50)_{13}}.$$

   (d)

$$p_{X_7}(j) = \begin{cases} \frac{(20)_7(30)_{j-7}}{(50)_j}\binom{j-1}{6} & \text{if } j \in \{7,8\dots 37\} \\ 0 & \text{else} \end{cases}$$

5. (a) increasing
   (b)

$$\mathbb{P}\left\{X_1 \le \frac{2}{3}\right\} = \mathbb{P}\left\{U \le \left(\frac{2}{3}\right)^{1/3}\right\} = \left(\frac{2}{3}\right)^{1/3}.$$

(c)

$$F_{X_1}(t) = \mathbb{P}\{U^3 \le t\} = \mathbb{P}\{U \le t^{1/3}\} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(d) X_1 is continuous and has density

$$f_{X_1}(t) = \begin{cases} \frac{1}{3}t^{-2/3} & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

(e) decreasing

(f)

$$\mathbb{P}\{X_2 \le 5\} = \mathbb{P}\left\{\frac{1}{U^3} \le 5\right\} = \mathbb{P}\left\{U \ge \frac{1}{5^{1/3}}\right\} = 1 - \frac{1}{5^{1/3}}.$$

(g) Since U takes values in $(0,1)$, X_2 takes values in $(1,\infty)$. Thus $F_{X_2}(t) = 0$ for $t \le 1$. For $t > 1$, we have that $F_{X_2}(t) = \mathbb{P}\left\{\frac{1}{U^3} \le t\right\} = \mathbb{P}\left\{U \ge t^{-1/3}\right\} = 1 - t^{-1/3}$. Thus

$$F_{X_2}(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - t^{-1/3} & \text{if } t \ge 1 \end{cases}$$

(h) X_2 is continuous and has density

$$f_{X_2}(t) = \begin{cases} \frac{1}{3}t^{-4/3} & \text{if } t > 1 \\ 0 & \text{else} \end{cases}$$

(i) decreasing

(j)

$$\mathbb{P}\{X_3 \le 7\} = \mathbb{P}\{\ln U \ge -35\} = \mathbb{P}\left\{U \ge e^{-35}\right\} = 1 - e^{-35}.$$

(k) Since U takes values in $(0,1)$, X_3 takes values in $(0,\infty)$. Thus $F_{X_3}(t) = 0$ for $t \le 0$. For $t > 0$, we have that $F_{X_2}(t) = \mathbb{P}\{\ln U \ge -5t\} = \mathbb{P}\{U \ge 1 - e^{-5t}\} = 1 - e^{-5t}$. Thus

$$F_{X_3}(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-5t} & \text{if } t \ge 0 \end{cases}$$

(l) X_3 is continuous and has density

$$f_{X_3}(t) = \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

6. (a)

$$\begin{aligned}\mathbb{P}\{X=5\} &= \sum_j \mathbb{P}\{X=5,\, Y=j\} = \sum_{j=5}^{\infty}\binom{j}{5}p^j(1-p)^{5-j}e^{-\lambda}\frac{\lambda^j}{j!}\\ &= \frac{e^{-\lambda}p^5}{5!}\sum_{j=5}^{\infty}\frac{(1-p)^{5-j}\lambda^j}{(5-j)!} = \frac{e^{-\lambda}(\lambda p)^5}{5!}\sum_{j=5}^{\infty}\frac{(\lambda(1-p))^{5-j}}{(5-j)!}\\ &= \frac{e^{-\lambda}(\lambda p)^5}{5!}\sum_{j=0}^{\infty}\frac{(\lambda(1-p))^{j}}{j!} = \frac{\exp\left[-\lambda+\lambda(1-p)\right](\lambda p)^5}{5!}\\ &= \frac{\exp\left[-\lambda p\right](\lambda p)^5}{5!}.\end{aligned}$$

(b)

$$p_X(i) = \begin{cases} e^{-\lambda p}\frac{(\lambda p)^i}{i!} & \text{if } i\in\{0,1\dots\}\\ 0 & \text{else}\end{cases}$$

(c) X is Poisson.

(d)

$$\mathbb{P}\{Y=7\} = \sum_i \mathbb{P}\{X=i,\, Y=7\} = \sum_{i=0}^{7}\binom{7}{i}p^i(1-p)^{7-i}e^{-\lambda}\frac{\lambda^7}{7!} = e^{-\lambda}\frac{\lambda^7}{7!}.$$

(e)

$$p_Y(j) = \begin{cases} e^{-\lambda}\frac{\lambda^j}{j!} & \text{if } j\in\{0,1\dots\}\\ 0 & \text{else}\end{cases}$$

(f) Y is Poisson

(g) They are not independent.

(h)

$$\begin{aligned}\mathbb{P}\{X=5|Y=7\} &= \frac{\mathbb{P}\{X=5,\, Y=7\}}{\mathbb{P}\{Y=7\}} = \frac{p_{X,Y}(5,7)}{p_Y(7)}\\ &= \frac{\binom{7}{5}p^i(1-p)^{7-5}e^{-\lambda}\frac{\lambda^7}{7!}}{e^{-\lambda}\frac{\lambda^7}{7!}} = \binom{7}{5}p^5(1-p)^2.\end{aligned}$$