1. 10 points Suppose that we want to arrange 15 textbooks on a shelf. There are
   • 6 math books
   • 5 history books
   • 4 chemistry books.

   (a) 5 points How many ways can we arrange the textbooks if we want to keep each subject together?

   (b) 5 points How many ways can we arrange the textbooks if we want a math book at each end?
Answers

1. (a) $6!5!4!3!$
   (b) $(6)_213!$
1. 10 points This is problem 36 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards.
   
   (a) 5 points What is the probability that both are aces?
   
   (b) 5 points What is the probability that both have the same face value?
1. Set $p \overset{\text{def}}{=} 1/\binom{52}{2}$.
   
   (a) $\binom{4}{2}p$.
   
   (b) $13\binom{4}{2}p$. 

R. Sowers
1. [10 points] This is problem 18 from the book. Suppose that two cards are randomly chosen from a deck of 52 cards. What is the probability that one of the cards is an ace and the other card is either a 10, jack, queen or king (i.e., what is the probability of a blackjack)?
1.

\[ 4 \times 16 / \binom{52}{2}. \]
1. [10 points] This is essentially problem 35 from the book. It is your birthday! Time to do some math. Mom or Dad have hidden a present for you, either upstairs or downstairs. With probability .6, the present was hidden by Mom. If Nom has hidden the present, it is upstairs with probability .7. Dad is equally likely to have hidden it upstairs or downstairs.

(a) [5 points] What is the probability that the present is upstairs?

(b) [5 points] If it is upstairs, what is the probability that it was hidden by Dad?
Answers

1. \( M = \{\text{Mom hid the present}\} \) and \( U = \{\text{present is upstairs}\} \).

We know that
\[
P(M) = .6, \quad P(U|M) = .7, \quad \text{and} \quad P(U|D) = .5.
\]

We then compute that
\[
P(U \cap M) = P(U|M)P(M) = .7 \times .6 = .42
\]
\[
P(U \cap D) = P(U|D)P(D) = .5 \times .4 = .20
\]

(a) \( P(U) = P(U \cap M) + P(U \cap D) = .42 + .2 = .62. \)

(b) \( P(D|U) = P(U \cap D)/P(U) = .2/.62. \)
1. 10 points Suppose we have an unfair coin with \( P\{H\} = p \) and \( P\{T\} = 1 - p \). We flip the coin a countable sequence of times. We are interested in the statistics of the third heads.

(a) 2 points Pick a specific configuration where the third heads is on the 9th toss.

(b) 2 points What is the probability of the configuration chosen in part a)?

(c) 3 points What is the probability that the third heads is on the 9th toss?

(d) 2 points For any integer \( n \geq 3 \), what is the probability that the third heads is on the \( n \)-th toss.
Answers

1. (a) $TTTHTHTTH$
   
   (b) $p^3(1 - p)^6$.
   
   (c) $\binom{6}{3}p^3(1 - p)^6$.
   
   (d) $\binom{n-1}{2}p^3(1 - p)^{n-3}$.
1. [10 points] A box has balls labelled $B_1$ through $B_{20}$. Pick three balls. Let $X$ be the number of the middle ball. In other words, if we pick balls $B_5$, $B_3$ and $B_{17}$, we order them and the middle ball is $B_5$, so $X = 5$.

(a) [2 points] Verbally describe what it means that $X = 5$.

(b) [2 points] Find $\mathbb{P}\{X = 5\}$.

(c) [2 points] Find $\mathbb{P}\{X = 9\}$.

(d) [2 points] Find the possible values of $X$.

(e) [2 points] Find the probability mass function of $X$; i.e., find $\mathbb{P}\{X = n\}$. 
1. (a) There is one ball among \( \{B_1, B_2 \ldots B_4\} \), we picked \( B_5 \), and there is one ball among \( \{B_5 \ldots B_{20}\} \).

(b) \( 4 \times (20 - 5)/(\binom{20}{3}) \).

(c) \( 8 \times (20 - 9)/(\binom{20}{3}) \).

(d) \( \{2, 3 \ldots 19\} \).

(e)

\[
p_X(n) = \begin{cases} 
\frac{(n-1)(20-n)}{\binom{20}{3}} & \text{if } n \in \{2, 3 \ldots 19\} \\
0 & \text{else}
\end{cases}
\]
1. **10 points** Let $X$ be geometric with parameter $p$; i.e.,

\[ p_X(n) = \begin{cases} (1 - p)^{n-1}p & \text{if } n \in \{1, 2, \ldots\} \\ 0 & \text{else} \end{cases} \]

Define the function $f(x) \overset{\text{def}}{=} \max\{x, 10\}$ for all $x \in \mathbb{R}$. Define $Y \overset{\text{def}}{=} f(X)$.

(a) **2 points** Graph $f$.

(b) **2 points** Compute $\mathbb{P}\{X = 5\}$.

(c) **2 points** Compute $\mathbb{P}\{X = 15\}$.

(d) **2 points** Compute $\mathbb{P}\{X = 10\}$.

(e) **2 points** Compute the probability mass function of $Y$. 

Answers

1. (a) Flat line at height 10, then up with slope 1.
   
   (b) \( \mathbb{P}\{Y = 5\} = 0. \)

   (c) \( \mathbb{P}\{Y = 15\} = \mathbb{P}\{X = 15\} = (1 - p)^{14}p. \)

   (d)

   \[
   \mathbb{P}\{Y = 10\} = \mathbb{P}\{X \leq 10\} = 1 - \mathbb{P}\{X \geq 11\} = 1 - \sum_{n=11}^{\infty} (1 - p)^{n-1}p = 1 - (1 - p)^{10}. 
   \]

   (e)

   \[
   p_Y(n) = \begin{cases} 
   1 - (1 - p)^{10} & \text{if } n = 10 \\
   (1 - p)^{n-1}p & \text{if } n \in \{11, 12 \ldots \} \\
   0 & \text{else} 
   \end{cases}
   \]
1. [10 points] Let \( X \) be a random variable with probability mass function

\[
p_X(n) \begin{cases} 
\frac{1}{12} & \text{if } n = 2 \\
\frac{2}{12} & \text{if } n = 1 \\
\frac{3}{12} & \text{if } n = 0 \\
\frac{5}{12} & \text{if } n = -1 \\
\frac{1}{12} & \text{if } n = -2 
\end{cases} 
\]

(a) [2 points] Compute \( \mathbb{E}[X] \)

(b) [2 points] Compute \( \mathbb{E}[X^2] \)

(c) [3 points] Compute the variance of \( X \).

(d) [3 points] Compute \( \mathbb{E}[e^{5X}] \). I want to see that you know what this means; I know that you can’t explicitly evaluate exponentials without a calculator.
Answers

1. (a) \[ E[X] = \frac{2 \times 1 + 1 \times 2 + 0 \times 3 + (-1) \times 5 + (-2) \times 1}{12} = \frac{-3}{12}. \]

(b) \[ E[X^2] = \frac{2^2 \times 1 + 1^2 \times 2 + 0^2 \times 3 + (-1)^2 \times 5 + (-2)^2 \times 1}{12} = \frac{15}{12}. \]

(c) \[ E[X^2] - E[X]^2 = \frac{15}{12} - \left(\frac{-3}{12}\right)^2 = \frac{15 \times 12 - 9}{144} = \frac{180 - 9}{144} = \frac{171}{144}. \]

(d) \[ E[e^{5X}] = \frac{e^{10} \times 1 + e^5 \times 2 + 1 \times 3 + e^{-5} \times 5 + e^{-10} \times 1}{12} = \frac{e^{10} + 2e^5 + 3 + 5e^{-5} + e^{-10}}{12}. \]
1. [10 points] Let $X$ be a continuous random variable with density

$$f_X(t) \overset{\text{def}}{=} \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Compute the moment generating function $\Phi(\theta) \overset{\text{def}}{=} E[e^{\theta X}]$ for $\theta < 5$. 

R. Sowers
1. For $\theta < 5$,

$$\Phi(\theta) = 5 \int_{t=0}^{\infty} \exp \left[-(5 - \theta)t\right] dt = \frac{5}{5 - \theta}$$
1. **10 points** Let $U$ be a Uniform$(0,1)$ continuous random variable; i.e., it has density

$$f_U(t) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Define the transformation

$$g(u) \overset{\text{def}}{=} \begin{cases} 1 - u & \text{if } u \leq \frac{2}{3} \\ u & \text{if } u > \frac{2}{3} \end{cases}$$

and define $X \overset{\text{def}}{=} g(U)$.

(a) **5 points** Compute $F_X$, the cumulative distribution function of $X$.

(b) **5 points** Compute $f_X$, the density of $X$. 

R. Sowers
Answers

1. (a) \( F_X(t) = 1 \) if \( t \geq 1 \) and \( F_X(t) = 0 \) for \( t < 0 \). For \( \frac{1}{3} \leq t < \frac{2}{3} \), we have that

\[
F_X(t) = \mathbb{P} \left\{ 1 - t \leq U \leq \frac{2}{3} \right\} = \frac{2}{3} - (1 - t) = t - \frac{1}{3}.
\]

For \( \frac{2}{3} \leq t < 1 \), we have that

\[
F_X(t) = \mathbb{P}\{1 - t \leq U \leq t\} = t - (1 - t) = 2t - 1.
\]

Thus

\[
F_X(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{1}{3} & \text{if } \frac{1}{3} \leq t < \frac{2}{3} \\
2t - 1 & \text{if } \frac{2}{3} \leq t < 1 \\
1 & \text{if } t \geq 1 
\end{cases}
\]

(b) Differentiate \( F_X \) to get that

\[
f_X(t) = \begin{cases} 
1 & \text{if } \frac{1}{3} < t < \frac{2}{3} \\
2 & \text{if } \frac{2}{3} < t < 1 \\
0 & \text{else}
\end{cases}
\]
1. 10 points $X$ be an $N(0, 1)$ Gaussian; i.e., a continuous random variable with density

$$f_X(t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \quad t \in \mathbb{R}$$

Set $Y \overset{\text{def}}{=} X + 7$.

(a) 5 points Compute $P\{Y \leq 13\}$ in terms of an integral of $f_X$.

(b) 5 points Compute the cumulative distribution function of $Y$ in terms of an integral of $f_X$. 

R. Sowers
1. (a)

\[ P\{Y \leq 13\} = P\{X + 7 \leq 13\} = P\{X \leq 6\} = \int_{s=-\infty}^{6} f_X(s) \, ds \]

(b) For \( t \in \mathbb{R} \)

\[ F_Y(t) = P\{X + 7 \leq t\} = P\{X \leq t - 7\} = \int_{s=-\infty}^{t-7} f_X(s) \, ds. \]
1. Evaluate the integral

\[ \int_{s=0}^{5} 0 \, ds. \]
Answers

1. 0.
Math 461, Section X13 and X14, Fall 2009
Exam 1, September 25

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don’t over-reduce your answers
Remember that I have to grade this; be neat

1. **45 points** This is an extension of Wednesday’s quiz. Suppose we have an unfair coin with \( P(H) = p \) and \( P(T) = 1 - p \). We flip the coin a countable sequence of times. We are interested in the statistics of the first and third heads.

   (a) **5 points** Pick a specific configuration where the third heads is on the 9th toss.

   (b) **5 points** What is the probability of the configuration chosen in part a)?

   (c) **5 points** What is the probability that the third heads is on the 9th toss?

   (d) **5 points** For any integer \( n \geq 3 \), what is the probability that the third heads is on the \( n \)-th toss.

   (e) **5 points** Pick a specific configuration where the first heads is on the 4th toss and the third heads is on the 9th toss.

   (f) **5 points** What is the probability of the configuration chosen in part e)?

   (g) **5 points** What is the probability that the first heads is on the 4th toss and the third heads is on the 9th toss.

   (h) **5 points** Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the 4th toss?

   (i) **5 points** Conditioned on the event that the third heads is on the 9th toss, what is the probability that the first heads is on the \( n \)-th toss, for each \( n \in \{1, 2 \ldots 7\} \).

2. **20 points** Suppose that

   \[
P(A) = .3 \quad P(B) = .6 \quad P((A \cup B)^c) = .28.
   \]

   (a) **5 points** Compute \( P(A \cup B) \).

   (b) **5 points** Compute \( P(A \cap B) \).

   (c) **5 points** Compute \( P(A \setminus B) \).

   (d) **5 points** Are \( A \) and \( B \) independent?

3. **21 points** Suppose that Jack, Betty, and Rick are in a team of 30 people. Two committees will be formed; committee \( A \) contains 7 people and committee \( B \) contains of 9 people. Assume that all committees are equally likely. Assume also that the committees can overlap.

   (a) **7 points** Find the probability that Jack and Rick are both on committee \( A \).

   (b) **7 points** Find the probability that Jack and Rick are both on committee \( A \) and Jack and Betty are on the same committee.

R. Sowers
(c) 7 points Find the probability that Jack and Betty are both on the same committee given that Jack and Rick are on committee A.

4. 14 points This question is motivated by question 3.19 in chapter 3. A “quit smoking” class is 40% male. A total of 30% of the women who took the class remained nonsmokers for a year. Similarly, a total of 20% of men who took the class remained nonsmokers for a year. A party was held at the end of the year for those who had quit for the year.

(a) 7 points What percentage of the original class was at the party?
(b) 7 points What percent of those attending the party were male?
Answers

1. (a) \((HTTTTTHHTH)\)
   (b) \((1 - p)^6 p^3\)
   (c) \(\left(\frac{8}{2}\right) (1 - p)^6 p^3\).
   (d) \(\left(\begin{pmatrix} n - 1 \\ 2 \end{pmatrix}\right) (1 - p)^{n-3} p^3\).
   (e) \((TTTHTHTTH)\)
   (f) \((1 - p)^6 p^3\).
   (g) \(4(1 - p)^6 p^3\).
   (h) \(\frac{4}{2}\).
   (i) \(\frac{8 - n}{2}\).

2. (a) \(\mathbb{P}(A \cup B) = 1 - \mathbb{P}((A \cup B)^c) = 1 - .28 = .72\).
   (b) \(\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = .3 + .6 - .72 = .9 - .72 = .18\).
   (c) \(\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = .3 - .18 = .12\).
   (d) Yes; \(\mathbb{P}(A \cap B) = .18 = .3 \times .6 = \mathbb{P}(A)\mathbb{P}(B)\).

3.

\[ C = \{\text{Jack and Rick are on committee A}\} \]
\[ D = \{\text{Jack and Betty are on the same committee}\}. \]

(a)
\[ \mathbb{P}(C) = \frac{\binom{28}{5}}{\binom{30}{5}} = \frac{\binom{28}{5}}{\binom{30}{5}} \]

(b) The challenge here is to write \(C \cap D\) as a disjoint union. Let’s partition the space based on whether or not Betty is on committee A. We have \(C \cap D = E_1 \cup E_2\) where
\[ E_1 = C \cap D \cap \{\text{Betty is on A}\} \]
\[ E_2 = C \cap D \cap \{\text{Betty is not on A}\}. \]

Note that
\( C \cap \{\text{Betty is on A}\} \subset D \)

so \(E_1 = \{\text{Jack, Rick and Betty are on A}\}\). Secondly, if Betty is not on committee A but she and Jack are on the same committee, then they must both be on B. In other words,
\[ E_2 = \{\text{Jack and Rick are on committee A, but Betty is not}\} \]
\[ \text{and Jack and Betty are on committee B}\}. \]
Then

\[ P(E_1) = \left( \frac{27}{4} \right) \left( \frac{30}{7} \right) \]
\[ P(E_2) = \left( \frac{27}{5} \right) \left( \frac{28}{7} \right) \left( \frac{30}{9} \right) \]

so

\[ P(C \cap D) = \left( \frac{27}{4} \right) \left( \frac{30}{7} \right) + \left( \frac{27}{5} \right) \left( \frac{28}{7} \right) \left( \frac{30}{9} \right) \]

Alternately, \( P(C \cap D) = P(C) - P(C \setminus D) \) and

\[ C \setminus D = \{ \text{Jack and Rick are on committee A and Betty is not on committee A} \}. \]

Here

\[ P(C \cap D) = \frac{28}{5} \left( \frac{30}{7} \right) - \frac{27}{5} \left( \frac{30}{7} \right) \]

(c)

\[ P(D \mid C) = \frac{27}{4} \left( \frac{30}{9} \right) + \frac{27}{5} \left( \frac{28}{7} \right) \left( \frac{30}{9} \right) \]

4.

\( M = \{ \text{men who were in the class} \} \quad Q = \{ \text{quit smoking for a year} \} \).

We have

\[ P(M) = .4 \quad P(Q \mid M^c) = .3 \quad P(Q \mid M) = .2. \]

We have that

\[ P(Q \cap M) = P(Q \mid M) P(M) = .2 \times .4 = .08 \]
\[ P(Q \cap M^c) = P(Q \mid M^c) P(M^c) = .3 \times .6 = .18. \]

(a)

\[ P(Q) = P(Q \cap M) + P(Q \cap M^c) = .08 + .18 = .26. \]

(b)

\[ P(M \mid Q) = \frac{P(Q \cap M)}{P(Q)} = \frac{.08}{.26}. \]
1. **28 points** Let $X$ be a random variable with probability mass function

\[ p_X(n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{15} & \text{if } n = 3 \\ \frac{2}{15} & \text{if } n = 2 \\ \frac{2}{15} & \text{if } n = 1 \\ \frac{3}{15} & \text{if } n = 0 \\ \frac{5}{15} & \text{if } n = -1 \\ \frac{2}{15} & \text{if } n = -2 \end{cases} \]

Define $Y = X^2$.

(a) **6 points** Compute $E[X]$

(b) **6 points** Compute $E[X^2]$

(c) **8 points** Compute $P\{Y = 4\}$

(d) **8 points** Compute $p_Y$, the probability mass function of $Y$.

2. **12 points** Assume that we have a coin with bias $P\{H\} = \frac{2}{10}$ (i.e., the probability of getting a heads is $\frac{2}{10}$). Flip the coin 10 times. Define the random variable

\[ X = \begin{cases} 1 & \text{if the first, second, and third tosses come up heads} \\ 0 & \text{else} \end{cases} \]

(a) **4 points** Compute $E[X]$

(b) **3 points** Compute $E[X^2]$

(c) **2 points** Compute $E[X^3]$

(d) **3 points** Compute the variance of $X$.

3. **19 points** Take a coin, and define $p \stackrel{\text{def}}{=} P\{\text{heads}\}$. Flip the coin until the third heads appears; let $X$ be the position of the third heads.

(a) **4 points** Write down a specific configuration such that $X = 7$.

(b) **4 points** Compute the probability of the configuration in part a.

(c) **5 points** Compute the probability that $X = 7$.

(d) **6 points** Compute the probability mass function for $X$. 

R. Sowers
4. **17 points** Suppose that $\mathbb{E}[X] = 2$ and the variance of $X$ is 5. Define $Y = 7X + 1$

(a) **5 points** Compute the standard deviation of $X$.

(b) **5 points** Compute $\mathbb{E}[X^2]$.

(c) **5 points** Compute $\mathbb{E}[Y]$.

(d) **2 points** Compute $\mathbb{E}[Y^2]$.

5. **21 points** Let $X$ be a random variable with moment generating function

$$
\Phi(\theta) \overset{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \begin{cases} 
\frac{5}{5-\theta} & \text{if } \theta < 5 \\
\infty & \text{else}
\end{cases}
$$

(a) **6 points** Compute $\mathbb{E}[X]$.

(b) **5 points** Compute $\mathbb{E}[X^2]$.

(c) **5 points** Compute $\mathbb{E}[X^3]$.

(d) **5 points** Compute the variance of $X$.

6. **3 points** Suppose that Timmy Typewriter is typesetting a book. On any given letter, he typically makes an error with probability .00007. We assume that the letters are independent, there are 400 letters on a page and 10 pages in a chapter. Use the Poisson approximation to approximate the probability that there will be at most 3 errors in the first chapter.
1. (a) \( \mathbb{E}[X] = \frac{3 \times 1 + 2 \times 2 + 1 \times 2 + 0 \times 3 + (-1) \times 5 + (-2) \times 2}{15} = 0. \)
(b) \( \mathbb{E}[X^2] = \frac{3^2 \times 1 + 2^2 \times 2 + 1^2 \times 2 + 0^2 \times 3 + (-1)^2 \times 5 + (-2)^2 \times 2}{15} = \frac{32}{15}. \)
(c) \( \mathbb{P}\{Y = 4\} = \mathbb{P}\{X = 2\} + \mathbb{P}\{X = -2\} = \frac{4}{15}. \)
(d) \( p_Y(n) = \begin{cases} \frac{1}{15} & \text{if } n = 9 \\ \frac{4}{15} & \text{if } n = 4 \\ \frac{7}{15} & \text{if } n = 1 \\ \frac{3}{15} & \text{if } n = 0 \end{cases} \)

2. Note that \( X = X^2 = X^3. \)
(a) \( \mathbb{E}[X] = 1 \times \left( \frac{2}{10} \right)^3 + 0 \times \left( 1 - \left( \frac{2}{10} \right)^3 \right) = \frac{8}{1000} \)
(b) \( \mathbb{E}[X^2] = \frac{8}{1000} \)
(c) \( \mathbb{E}[X^3] = \frac{8}{1000} \)
(d) \( \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{8}{1000} - \left( \frac{8}{1000} \right)^2 = \frac{7936}{10^5}. \)

3. (a) \( TTHHTTH \)
(b) \( p^3(1 - p)^4. \)
(c) \( \binom{6}{2} p^3(1 - p)^4. \)
(d) \( p_X(n) = \begin{cases} \binom{n-1}{2} p^3(1 - p)^{n-3} & \text{if } n \in \{3, 4 \ldots \} \\ 0 & \text{else} \end{cases} \)

4. (a) \( \sqrt{5}. \)
(b) \( \mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = 5 + 4 = 9. \)
(c) \( \mathbb{E}[Y] = 7\mathbb{E}[X] + 1 = 7 \times 2 + 1 = 15. \)
(d) \( \mathbb{E}[Y^2] = \mathbb{E}[49X^2 + 14X + 1] = 49\mathbb{E}[X^2] + 14\mathbb{E}[X] + 1 = 49 \times 9 + 14 \times 2 + 1 = 470. \)

5. For \( \theta < 5, \)
\[ \Phi'(\theta) = \frac{5}{(5 - \theta)^2} \quad \Phi^{(2)}(\theta) = \frac{2 \times 5}{(5 - \theta)^3} \quad \Phi^{(2)}(\theta) = \frac{3! \times 5}{(5 - \theta)^3} \]
(a) \( \mathbb{E}[X] = \Phi'(0) = \frac{1}{5}. \)
(b) \( \mathbb{E}[X^2] = \Phi^{(2)}(0) = \frac{2}{25}. \)
(c) \( \mathbb{E}[X^3] = \Phi^{(3)}(0) = \frac{6}{125}. \)
(d) \( \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{25} - \frac{1}{25} = \frac{1}{25}. \)
6. We assume that the errors are Poisson distributed with parameter \( \lambda = 0.0007 \times 4000 = 2.8 \). Thus

\[
P\{X \leq 3\} = \sum_{n=0}^{3} e^{-2.8} \frac{(2.8)^n}{n!} = e^{-2.8} \left\{ 1 + 2.8 + \frac{(2.8)^2}{2} + \frac{(2.8)^3}{6} \right\}
\]
Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don’t over-reduce your answers
Remember that I have to grade this; be neat
For the purpose of this exam, we will define

\[ \Phi(t) \overset{\text{def}}{=} \int_{s=-\infty}^{t} \exp\left[\frac{-s^2}{2}\right] ds \quad t \in \mathbb{R} \]

1. **15 points** Let \( X \) be a random variable with cumulative distribution function

\[ F_X(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{t}{2} & \text{if } 0 \leq t < 1 \\
\frac{1}{2} + \frac{t-1}{5} & \text{if } 1 \leq t < 2 \\
\frac{5}{6} & \text{if } 2 \leq t < 3 \\
1 & \text{if } t \geq 3 
\end{cases} \]

(a) **5 points** Compute \( \mathbb{P}\{X = 1\} \).
(b) **5 points** Compute \( \mathbb{P}\{X = 2\} \).
(c) **5 points** Compute \( \mathbb{P}\{X < 3\} \).

2. **31 points** Let \( X \) be a continuous random variable with density function

\[ f_X(t) = \begin{cases} 
e^{-t} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases} \]

(a) **5 points** Compute \( \mathbb{P}\{X \geq 7\} \).
(b) **5 points** Compute \( \mathbb{P}\{X \geq 3\} \).
(c) **5 points** Compute \( \mathbb{P}\{X \geq 7 | X \geq 3\} \).

Define \( Y \overset{\text{def}}{=} X^2 \)

(d) **5 points** Compute \( \mathbb{P}\{Y \leq 4\} \).
(e) **5 points** Compute the cumulative distribution function of \( Y \).
(f) **6 points** Is \( Y \) continuous? If so, compute its density. If not, state why it is not continuous.

3. **54 points** Let \( X \) be a Gaussian with mean 1 and variance 4. Let \( N \) be a Gaussian with mean 0 and variance 1.

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(a) 5 points Write $X$ in terms of $N$.

(b) 5 points Compute $\mathbb{P}\{X \geq 3\}$ using the attached table.

(c) 6 points Compute $\mathbb{P}\{|X| \geq 3\}$ using the attached table.

(d) 5 points What is the density of $X$?

Define a function
\[
\varphi(z) \overset{\text{def}}{=} \begin{cases} 
-1 & \text{if } z \leq 0 \\
2z & \text{if } 0 < z < 3 \\
6 & \text{if } z \geq 3
\end{cases}
\]

(e) 5 points Graph $\varphi$.

Define $Y \overset{\text{def}}{=} \varphi(X)$. As necessary, answer the following questions in terms of $\Phi$, not the table.

(f) 5 points Compute $\mathbb{P}\{Y \leq -2\}$

(g) 6 points Compute $\mathbb{P}\{Y \leq 3\}$

(h) 5 points Compute $\mathbb{P}\{Y \leq 7\}$

(i) 7 points What is the cumulative distribution function of $Y$ (in terms of $\Phi$ as needed).

(j) 5 points Is $Y$ a continuous random variable? If not, why?
1. (a) 
\[ P\{X = 1\} = F_X(1) - F_X(1^-) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \]

(b) 
\[ P\{X = 2\} = F_X(2) - F_X(2^-) = \frac{5}{6} - \frac{7}{10} = \frac{4}{30} = \frac{2}{15} \]

(c) \[ P\{X < 3\} = F_X(3^-) = \frac{5}{6} \]

2. (a) 
\[ P\{X \geq 7\} = \int_{t=7}^{\infty} f_X(t) dt = \int_{t=7}^{\infty} e^{-t} dt = e^{-7} \]

(b) 
\[ P\{X \geq 3\} = \int_{t=3}^{\infty} f_X(t) dt = \int_{t=3}^{\infty} e^{-t} dt = e^{-3} \]

(c) 
\[ P\{X \geq 7|X \geq 3\} = \frac{P\{X \geq 7, X \geq 3\}}{P\{X \geq 3\}} = \frac{P\{X \geq 7\}}{P\{X \geq 3\}} = \frac{e^{-7}}{e^{-3}} = e^{-4} \]

(d) 
\[ P\{Y \leq 4\} = P\{|X| \leq 2\} = \int_{t=0}^{2} e^{-t} dt = 1 - e^{-2} \]

(e) 
\[ F_Y(t) = \begin{cases} 
0 & \text{if } t < 0 \\
1 - e^{-\sqrt{t}} & \text{if } t \geq 0 
\end{cases} \]

(f) \(Y\) is continuous and 
\[ f_Y(t) = \begin{cases} 
0 & \text{if } t < 0 \\
\frac{e^{-\sqrt{t}}}{2\sqrt{t}} & \text{if } t > 0 
\end{cases} \]

3. (a) \(X = 2N + 1\).

(b) 
\[ P\{X \geq 3\} = P\{2N + 1 \geq 3\} = P\{N \geq 1\} \\
= 1 - P\{N < 1\} = 1 - \Phi(1) = 1 - .8413 = .1587 \]

(c) 
\[ P\{|X| \geq 3\} = P\{X \leq -3\} + P\{X \geq 3\} = P\{2N + 1 \leq -3\} + P\{2N + 1 \geq 3\} \\
= P\{N \leq -2\} + P\{N \geq 1\} = P\{N \geq 2\} + P\{N \geq 1\} \\
= 1 - P\{N \leq 2\} + 1 - P\{N \leq 1\} = 2 - \Phi(2) - \Phi(1) = 2 - .9772 - .8413 = .1815 \]

(d) 
\[ f_X(t) = \frac{\exp \left[ -\frac{(t-1)^2}{8} \right]}{\sqrt{8\pi}} \quad t \in \mathbb{R} \]

(e)
(f) \( P\{Y \leq -2\} = P(\emptyset) = 0 \)

(g) 
\[
P\{Y \leq 3\} = P\left\{X \leq \frac{3}{2}\right\} = P\left\{2N + 1 \leq \frac{3}{2}\right\} = P\left\{N \leq \frac{1}{4}\right\} = \Phi\left(\frac{1}{4}\right)
\]

(h) \( P\{Y \leq 7\} = 1 \).

(i) 
\[
F_Y(t) = \begin{cases} 
0 & \text{if } t < -1 \\
\Phi\left(-\frac{1}{2}\right) & \text{if } -1 \leq t < 0 \\
\Phi\left(\frac{t/2-1}{2}\right) & \text{if } 0 \leq t < 6 \\
1 & \text{if } t \geq 6
\end{cases}
\]

Note that \( \Phi\left(-\frac{1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right) \).

(j) No; it has a jump at \(-1\) and \(6\).
1. **20 points** This is question 7 on page 16 of the book.
   (a) **5 points** How many ways can we seat 3 boys and 3 girls in a row?
   (b) **5 points** How many ways can we seat 3 boys and 3 girls in a row if the boys and girls are each to sit together?
   (c) **5 points** How many ways can we seat 3 boys and 3 girls in a row if only the boys must sit together?
   (d) **5 points** How many ways can we seat 3 boys and 3 girls in a row if no two people of the same gender are allowed to sit together?

2. **15 points** At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.
   (a) **10 points** Randomly select a computer science student. What is the probability that it is a woman?
   (b) **5 points** In this data, is gender independent of the decision to study computer science?

3. **12 points** Let \( X \) be a random variable with probability mass function

\[
p(j) = \begin{cases} 
\frac{2}{9} & \text{if } j = -1 \\
\frac{1}{9} & \text{if } j = 0 \\
\frac{2}{9} & \text{if } j = 3 \\
\frac{4}{9} & \text{if } j = 7 \\
0 & \text{else}
\end{cases}
\]

   (a) **6 points** Compute \( E[X] \).
   (b) **6 points** Compute \( E[\min\{X, 1\}] \).

4. **40 points** We will here develop some ideas relating to the negative hypergeometric distribution. This is similar in some ways to a negative binomial, and you might keep those calculations in mind.

Suppose that we have a box containing 20 black balls \( \{B_1, B_2 \ldots B_{20}\} \) and 30 red balls \( \{R_1, R_2 \ldots R_{30}\} \). Let’s pick balls out of the box, one by one.

Let \( X_7 \) be the first time that we have 7 black balls.
(a) 10 points Write a specific configuration where \( X_7 = 13 \)

(b) 10 points Compute the probability of this configuration.

(c) 10 points Compute the probability that \( P\{X_7 = 13\} \).

(d) 10 points What is the probability mass function of \( X_7 \)?

5. 66 points Let \( U \) be a continuous random variable which is uniformly distributed on \((0, 1)\); i.e., it has density \( f_U(t) = \begin{cases} 1 & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases} \)

Define \( \varphi_1(u) \overset{\text{def}}{=} u^3 \) for \( u \in (0, 1) \), and define \( X_1 \overset{\text{def}}{=} \varphi_1(U) \).

(a) 5 points Graph \( \varphi_1 \)

(b) 7 points Compute \( P\{X_1 \leq \frac{2}{3}\} \).

(c) 5 points Compute the cumulative of \( X_1 \).

(d) 4 points Does \( X_1 \) have a density? If so, compute it. If not, explain why.

Define \( \varphi_2(u) \overset{\text{def}}{=} 1/u^3 \) for \( u \in (0, 1) \), and define \( X_2 \overset{\text{def}}{=} \varphi_2(U) \).

(e) 5 points Graph \( \varphi_2 \)

(f) 7 points Compute \( P\{X_2 \leq 5\} \).

(g) 5 points Compute the cumulative of \( X_2 \).

(h) 4 points Does \( X_2 \) have a density? If so, compute it. If not, explain why.

Define \( \varphi_3(u) \overset{\text{def}}{=} -\frac{1}{3} \ln u \) for \( u \in (0, 1) \), and define \( X_3 \overset{\text{def}}{=} \varphi_3(U) \).

(i) 5 points Graph \( \varphi_3 \)

(j) 7 points Compute \( P\{X_3 \leq 7\} \).

(k) 5 points Compute the cumulative of \( X_3 \).

(l) 4 points Does \( X_3 \) have a density? If so, compute it. If not, explain why.

(m) 3 points \( X_3 \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poission, or uniform. Which one?

6. 47 points Fix two parameters \( p \in (0, 1) \) and \( \lambda > 0 \). Consider two discrete random variables \( X \) and \( Y \) with joint probability mass function

\[
p_{X,Y}(i,j) = \begin{cases} \binom{j}{i}p^i(1-p)^{j-i}e^{-\lambda \frac{j}{i}} & \text{if } j \in \{0,1\ldots\} \text{ and } i \in \{0,1\ldots j\} \\ 0 & \text{else} \end{cases}
\]

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(as usual, we define \( \binom{n}{0} \defeq 1 \)). In other words,

\[
P\{X = 2, Y = 7\} = \binom{7}{2} p^2 (1 - p)^5 e^{-\lambda} \frac{\lambda^7}{7!}.
\]

This is some combination of binomial and Poisson distributions, of course.

(a) 10 points Compute \( p_X(5) \defeq P\{X = 5\} \). Hint: recall that \( \binom{j}{i} = \frac{j!}{i!(j-i)!} \). Recall also the Taylor formula for the exponential.

(b) 5 points Compute the probability mass function of \( X \).

(c) 3 points \( X \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poisson, or uniform. Which one?

(d) 10 points Compute \( p_Y(7) \defeq P\{Y = 7\} \) Hint: recall the binomial theorem.

(e) 5 points Compute the probability mass function of \( Y \).

(f) 3 points \( Y \) is one of the following types of random variables: Bernoulli, binomial, exponential, Gaussian, geometric, hypergeometric, negative binomial, Poisson, or uniform. Which one?

(g) 4 points Are \( X \) and \( Y \) independent?

(h) 7 points Compute \( P\{X = 5|Y = 7\} \).
Answers

1. (a) 6!
   (b) $2 \times 3! \times 3!$ (the 2 comes from which gender is left-most)
   (c) $4! \times 3!$ (group the boys as one unit, and then order within the unit)
   (d) $2 \times 3! \times 3!$ (the two comes from which gender is left-most).

2. $W = \{\text{woman}\}$ and $C = \{\text{computer science}\}$.
   (a) $P(W|C) = \frac{P(W \cap C)}{P(C)} = \frac{7}{10} = .7$.
   (b) No; $P(W|C) = .7 \neq .54 = P(W)$.

3. (a) 
   $E[X] = \frac{(-1) \times 2 + 0 \times 1 + 3 \times 2 + 7 \times 4}{9} = \frac{32}{9}$.
   (b) 
   $E[\min\{X, 1\}] = \sum_j \min\{j, 1\}p(j)$
   
   $= \min\{-1, 1\} \frac{2}{9} + \min\{0, 1\} \frac{1}{9} + \min\{3, 1\} \frac{2}{9} + \min\{7, 1\} \frac{4}{9}$
   
   $= (-1) \frac{2}{9} + (0) \frac{1}{9} + (1) \frac{2}{9} + (1) \frac{4}{9} = \frac{-2 + 0 + 2 + 4}{9} = \frac{4}{9}$.

4. (a) $(B_4, B_2, R_9, R_{11}, R_3, B_3, R_7, B_{20}, B_{15}, R_{30}, B_8, R_1, B_1)$
   (b) 
   $\frac{1}{(50)_{13}}$.
   (c) 
   $\binom{12}{6} \frac{(20)_{7}(30)_{6}}{(50)_{13}}$.
   (d) 
   $p_{X_7}(j) = \begin{cases} 
   \frac{(20)_{7}(30)_{j \leq 7}}{(50)_{13} \binom{6}{j}} & \text{if } j \in \{7, 8 \ldots 37\} \\
   0 & \text{else} 
   \end{cases}$

5. (a) increasing
   (b) 
   $\mathbb{P}\left\{ X_1 \leq \frac{2}{3} \right\} = \mathbb{P}\left\{ U \leq \left(\frac{2}{3}\right)^{1/3} \right\} = \left(\frac{2}{3}\right)^{1/3}$. 

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(c) \[ F_{X_1}(t) = \mathbb{P}\{U^3 \leq t\} = \mathbb{P}\{U \leq t^{1/3}\} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} \]

(d) \( X_1 \) is continuous and has density

\[ f_{X_1}(t) = \begin{cases} \frac{1}{3}t^{-2/3} & \text{if } t \in (0, 1) \\ 0 & \text{else} \end{cases} \]

(e) decreasing

(f) \[ \mathbb{P}\{X_2 \leq 5\} = \mathbb{P}\left\{ \frac{1}{U^3} \leq 5 \right\} = \mathbb{P}\left\{ U \geq \frac{1}{5^{1/3}} \right\} = 1 - \frac{1}{5^{1/3}}. \]

(g) Since \( U \) takes values in \((0, 1)\), \( X_2 \) takes values in \((1, \infty)\). Thus \( F_{X_2}(t) = 0 \) for \( t \leq 1 \). For \( t > 1 \), we have that \( F_{X_2}(t) = \mathbb{P}\{U \leq t^{1/3}\} = \mathbb{P}\{U \geq t^{-1/3}\} = 1 - t^{-1/3} \). Thus

\[ F_{X_2}(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - t^{-1/3} & \text{if } t \geq 1 \end{cases} \]

(h) \( X_2 \) is continuous and has density

\[ f_{X_2}(t) = \begin{cases} \frac{1}{3}t^{-4/3} & \text{if } t > 1 \\ 0 & \text{else} \end{cases} \]

(i) decreasing

(j) \[ \mathbb{P}\{X_3 \leq 7\} = \mathbb{P}\{\ln U \geq -35\} = \mathbb{P}\{U \geq e^{-35}\} = 1 - e^{-35}. \]

(k) Since \( U \) takes values in \((0, 1)\), \( X_3 \) takes values in \((0, \infty)\). Thus \( F_{X_3}(t) = 0 \) for \( t \leq 0 \). For \( t > 0 \), we have that \( F_{X_3}(t) = \mathbb{P}\{\ln U \geq -5t\} = \mathbb{P}\{U \geq 1 - e^{-5t}\} = 1 - e^{-5t} \). Thus

\[ F_{X_3}(t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-5t} & \text{if } t \geq 0 \end{cases} \]

(l) \( X_3 \) is continuous and has density

\[ f_{X_3}(t) = \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases} \]
6. (a) 

\[ \mathbb{P}\{X = 5\} = \sum_j \mathbb{P}\{X = 5, Y = j\} = \sum_{j=5}^{\infty} \left(\begin{array}{c} j \\ 5 \end{array}\right) p^j (1-p)^{5-j} e^{-\lambda \frac{j}{j!}} \]

\[ = \frac{e^{-\lambda} p^5}{5!} \sum_{j=5}^{\infty} \frac{(1-p)^{5-j} \lambda^j}{(5-j)!} = \frac{e^{-\lambda} \lambda^5}{5!} \sum_{j=5}^{\infty} \frac{(\lambda(1-p))^{5-j}}{(5-j)!} \]

\[ = \frac{e^{-\lambda} \lambda^5}{5!} \sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!} = \exp \left[ -\lambda + \lambda(1-p) \right] \frac{(\lambda p)^5}{5!} \]

\[ = \exp \left[ -\lambda p \right] \frac{(\lambda p)^5}{5!}. \]

(b) 

\[ p_X(i) = \begin{cases} e^{-\lambda p} \frac{(\lambda p)^i}{i!} & \text{if } i \in \{0, 1, \ldots\} \\ 0 & \text{else} \end{cases} \]

(c) \( X \) is Poisson.

(d) 

\[ \mathbb{P}\{Y = 7\} = \sum_i \mathbb{P}\{X = i, Y = 7\} = \sum_{i=0}^{7} \left(\begin{array}{c} 7 \\ i \end{array}\right) p^i (1-p)^{7-i} e^{-\lambda \frac{7}{7!}} = e^{-\lambda} \frac{\lambda^7}{7!}. \]

(e) 

\[ p_Y(j) = \begin{cases} e^{-\lambda} \frac{\lambda^j}{j!} & \text{if } j \in \{0, 1, \ldots\} \\ 0 & \text{else} \end{cases} \]

(f) \( Y \) is Poisson

(g) They are not independent.

(h) 

\[ \mathbb{P}\{X = 5|Y = 7\} = \frac{\mathbb{P}\{X = 5, Y = 7\}}{\mathbb{P}\{Y = 7\}} = \frac{p_{X,Y}(5,7)}{p_Y(7)} = \frac{\left(\begin{array}{c} 5 \\ 5 \end{array}\right) p^5 (1-p)^{7-5} e^{-\lambda \frac{7}{7!}}}{e^{-\lambda} \frac{\lambda^7}{7!}} = \left(\begin{array}{c} 7 \\ 5 \end{array}\right) p^5 (1-p)^2. \]