Math 461, Section F13, Fall 2008
Quiz 1, September 3 **Name:** ____________________

1. 10 points Suppose that we have 30 people in a room. How many ways can we form two disjoint committees, where the first committee has 5 people and the second one has 6 people?

Answers

1.

$$\binom{30}{5} \times \binom{25}{6}.$$

Math 461, Section F13, Fall 2008
Quiz 2, September 12 **Name:** ____________________

1. 10 points This is question 18 in the book.

   Two cards are randomly selected from a standard deck of cards. What is the probability that one of them is an ace and the other is either a ten, jack, queen, or king?

Answers

1.

$$\frac{4 \times 16}{\binom{52}{2}}$$

Math 461, Section F13, Fall 2008
Quiz 3, October 1 **Name:** ______________________

1. 10 points This is similar to question 20 on page 113.

   At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.

   Randomly select a computer science student. What is the probability that it is a woman?

Answers

1.

$$W = \{\text{woman}\} \qquad \text{and} \qquad C = \{\text{computer science}\}.$$

Then

$$\mathbb{P}(W|C) = \frac{\mathbb{P}(W \cap C)}{\mathbb{P}(C)} = \frac{7}{10} = .7.$$

Math 461, Section F13, Fall 2008
Quiz 4, October 8 **Name:** ____________________

1. 10 points Assume that we have a fair die. Consider a random variable X such that

$$X(1) = X(2) = 4, \quad X(3) = 0, \quad X(4) = -1, \quad X(5) = 4, \quad X(6) = 0.$$

Compute the probability mass function p_X of X.

Answers

1.

$$p_X(j) = \begin{cases} \frac{3}{6} & \text{if } j = 4 \\ \frac{2}{6} & \text{if } j = 0 \\ \frac{1}{6} & \text{if } j = -1 \\ 0 & \text{else} \end{cases}$$

Math 461, Section F13, Fall 2008
Quiz 5, October 13 **Name:** ______________________

1. [10 points] Assume that we have a fair die. Consider a random variable X such that

$$X(1) = X(2) = 4, \quad X(3) = 0, \quad X(4) = -1, \quad X(5) = 4, \quad X(6) = 0.$$

   (a) [5 points] Compute $\mathbb{E}[X]$

   (b) [5 points] Compute $\mathbb{E}[X^2]$

Answers

1. (a)

$$\mathbb{E}[X] = \frac{4+4+(-1)+4}{6} = \frac{11}{6}.$$

   (b)

$$\mathbb{E}[X^2] = \frac{4^2+4^2+(-1)^2+4^2}{6} = \frac{49}{6}.$$

Math 461, Section F13, Fall 2008
Quiz 6, October 27 **Name:** ____________________

1. [10 points] This is essentially question 38 on page 192. Let X be a random variable such that $\mathbb{E}[X] = 2$ and $\mathbb{E}[X^2] = 6$. Set $Y \stackrel{\text{def}}{=} 2X + 3$.
   (a) [5 points] Compute $\mathbb{E}[Y]$
   (b) [5 points] Compute the variance of Y.

Answers

1. (a)
$$\mathbb{E}[Y] = 2\mathbb{E}[X] + 3 = 2 \times 2 + 3 = 7.$$

   (b)
$$\mathbb{E}[Y^2] = \mathbb{E}[(2X+3)^2] = 4\mathbb{E}[X^2] + 12\mathbb{E}[X] + 9 = 4 \times 6 + 12 \times 2 + 9 = 57$$

   so
$$\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 57 - 49 = 8.$$

Math 461, Section F13, Fall 2008
Quiz 7, November 9 **Name:** ______________________

1. 10 points Let X be a Uniform(0,1) random variable. Define $Y \stackrel{\text{def}}{=} X^{1/2}$.

   (a) 5 points Compute the cumulative distribution function F_Y of Y.

   (b) 5 points Compute the density of Y if it exists; if it doesn't exist, explain why.

1. (a) We have that

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \mathbb{P}\{X^{1/2} \le t\} = \begin{cases} 0 & \text{if } t < 0 \\ \mathbb{P}\{X \le t^2\} & \text{if } t \ge 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 1 \\ t^2 & \text{if } 0 < t \le 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(b) F_Y is continuous and has density

$$f_Y(t) = \begin{cases} 2t & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 461, Section F13, Fall 2008

Quiz 8, November 10 **Name:** ____________________

1. [10 points] Let X be a Uniform(0,1) random variable. Define $Y \stackrel{\text{def}}{=} X^{-1/2}$.

   (a) [2 points] Graph $x \mapsto x^{-1/2}$ for $0 < x < 1$.

   (b) [6 points] Compute the cumulative distribution function F_Y of Y.

   (c) [2 points] Compute the density of Y if it exists; if it doesn't exist, explain why.

Answers

1. (a) The graph decreases from ∞ at $x=0$ to 1 at $x=1$.

   (b) We have that

$$F_Y(t)=\mathbb{P}\{Y\le t\}=\mathbb{P}\{X^{-1/2}\le t\}=\begin{cases}0 & \text{if } t<1\\ \mathbb{P}\{X\ge 1/t^2\} & \text{if } t\ge 1\end{cases}$$
$$=\begin{cases}0 & \text{if } t<1\\ 1-\frac{1}{t^2} & \text{if } t\ge 1\end{cases}$$

   (c) F_Y is continuous and has density

$$f_Y(t)=\begin{cases}2t^{-3} & \text{if } t>1\\ 0 & \text{else}\end{cases}$$

Math 461, Section F13, Fall 2008
Quiz 9, November 18 **Name:** ____________________

1. [10 points] Let X be a Gaussian random variable with mean 0 and variance 4. Define $Y \stackrel{\text{def}}{=} X^2$.
   (a) [7 points] Compute the cumulative distribution function of Y.
   (b) [3 points] Compute the density of Y if it exists; if not, explain why the density doesn't exist.

2. (a) We have that $F_Y(t) = \mathbb{P}\{Y \le t\} = \mathbb{P}\{X^2 \le t\}$. If $t < 0$, then $F_Y(t) = 0$. If $t \ge 0$, then

$$F_Y(t) = \mathbb{P}\{-\sqrt{t} \le X \le \sqrt{t}\} = F_X(\sqrt{t}) - F_X(-\sqrt{t}-) = F_X(\sqrt{t}) - F_X(-\sqrt{t}) = \int_{z=-\sqrt{t}}^{\sqrt{t}} \frac{e^{-z^2/8}}{\sqrt{8\pi}} dz.$$

(b) F_Y is continuous and differentiable (except at 0). If $t < 0$, then $f_Y(t) = 0$. If $t > 0$, then

$$f_Y(t) = \frac{1}{2\sqrt{t}} \left\{ f_X(\sqrt{t}) + f_X(-\sqrt{t}) \right\} = \frac{1}{\sqrt{8\pi t}} \exp\left[-\frac{t}{8}\right]$$

Math 461, Section F13, Fall 2008
Quiz 10, December 3 **Name:** ______________________

1. [10 points] Let the lifetime X of a lightbulb have hazard function

$$h_X(t) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } t < 1 \\ \frac{2}{t} & \text{if } t \geq 1 \end{cases}$$

Compute the cumulative distribution function of X.

Answers

1. Set $G_X(t) = \mathbb{P}\{X > t\} = 1 - F_X(t)$. Then

$$\dot{G}_X(t) = -f_X(t) = -h_X(t)G(t).$$

Also, $G_X(0) = 1 - F_X(0) = 1$, so $G_X(t) = \exp\left[-\int_{s=0}^t h_X(s)ds\right]$. Thus $G_X(t) = 1$ for $t < 1$. For $t > 1$,

$$G_X(t) = \exp\left[-\int_{s=1}^t \frac{2}{s}ds\right] = \exp\left[-2\ln t\right] = t^{-2}.$$

Thus

$$F_X(t) = 1 - G_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - \frac{1}{t^2} & \text{if } t \geq 1 \end{cases}$$

Math 461, Section F13, Fall 2008

Quiz 11, December 5 **Name:** ____________________

1. 10 points Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} te^{-st} & \text{if } s > 0 \text{ and } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Compute f_Y, the density of Y.

Answers

1.

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{s=0}^{\infty} te^{-st}ds & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases} = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Math 461, Section F13, Fall 2008
Quiz 12, December 8 **Name:** ____________________

1. 10 points Evaluate the following expression: $12 - 12$.

Answers

1. 0.

Math 461, Section F13, Fall 2008
Exam 1, September 19

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. [25 points] This is essentially 20 on page 17. Assume that Ted Thinksalot has 50 friends. He wants to have a party with 10 people.
   (a) [15 points] Ted knows that Pat and Pete detest each other for some reason. How many ways can he issue the invitations so that Pat and Pete won't both be invited?
   (b) [10 points] Ted now learns that the reason why Pete and Pat don't like each other is that Pete is now going out with Paula, who used to be Pat's girlfriend. Pete and Paula are now inseparable, and will only come together. How many ways can Ted issue the invitations now?

2. [25 points] This is essentially question 37 on page 59. Professor Dork gives his class a study guide of 20 problems in preparation for an exam. He says that the exam will consist of 8 questions randomly chosen (with equal likelihood) from the 20. Suppose that you know how to do 12 of the problems in the study guide.
   (a) [15 points] What is the probability that you will get 100% on the exam?
   (b) [10 points] What is the probability that you will get exactly 87.5% on the exam?

3. [25 points] This is essentially question 47 and 48 on page 60. Assume that there are 21 people in a room.
   (a) [15 points] What is the probability that no two people have birthdays in the same week?
   (b) [10 points] What is the probability that, among the 12 months of the year, there are 5 months containing 3 birthdays each, and 1 month which contains 6 birthdays?

4. [25 points] This is question 52 on page 60. A closet contains 10 pairs of shoes. If 8 shoes are selected at random, what is the probability that there will be
   (a) [15 points] no complete pair?
   (b) [10 points] exactly one complete pair?

Answers

1. (a)

$$\binom{48}{10} + 2\binom{48}{9}.$$

(b)

$$\binom{47}{10} + \binom{47}{9} + \binom{47}{8}.$$

2. (a)

$$\frac{\binom{12}{8}}{\binom{20}{8}}.$$

(b)

$$\frac{\binom{12}{7}\binom{8}{1}}{\binom{20}{8}}.$$

3. (a)

$$\frac{(52)_{21}}{52^{21}}.$$

(b)

$$\frac{\binom{21}{3\,3\,3\,3\,3\,6}(12)_6}{12^{21}}.$$

4. (a)

$$\frac{\binom{10}{8}2^8}{\binom{20}{8}}.$$

(b)

$$\frac{10\binom{9}{6}2^6}{\binom{20}{8}}.$$

Math 461, Section F13, Fall 2008
Exam 2, October 24

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. [22 points] This is essentially question 28 on page 114 Take a deck of cards. Turn the cards over one by one until an ace appears. Given that the first ace is the 15th card, what is the probability that the 16th card will be
   (a) [11 points] The three of diamonds
   (b) [11 points] The ace of spades

2. [22 points] This is essentially question 47 on page 116. Do NOT attempt to carry out the calculations to the bitter end; stop at the point where you would reach for a calculator (which is not allowed on the exam anyways). A box has 6 white balls and 11 black balls. Toss a fair die, and pick out of the box the number of balls show on the die.
   (a) [11 points] What is the probability that all balls are white?
   (b) [11 points] Given that all balls are white, what is the probability that the die showed a four?

3. [22 points] This is question 1 on page 187. A box contains 8 white, 4 black, and 2 orange balls. We randomly choose 2 balls from the box. Suppose that we win \$2 for each black ball, and we lose \$1 for each white ball. Let X denote our winnings.
   (a) [11 points] What are the possible values of X?
   (b) [11 points] What is the probability mass function for X?

4. [34 points] This is essentially question 35 on page 192. A box contains 5 white and 5 black marbles. Randomly pick two. You win \$2 if both marbles are of the same color and you lose \$1 if the marbles are of different color. Let X be your winnings.
   (a) [11 points] Compute the probability mass function of X.
   (b) [11 points] Compute $\mathbb{E}[X]$.
   (c) [12 points] Compute the variance of X.

Answers

1. (a)

$$\frac{\frac{(47)_{14}\times 4\times 1}{(52)_{16}}}{\frac{(48)_{14}\times 4}{(52)_{15}}} = \frac{34}{48\times 37} = \frac{17}{24\times 37}$$

   (b)

$$\frac{\frac{(48)_{14}\times 3\times 1}{(52)_{16}}}{\frac{(48)_{14}\times 4}{(52)_{15}}} = \frac{3}{4\times 37}$$

2. (a)

$$\sum_{i=1}^{6} \frac{\binom{6}{i}}{\binom{17}{i}} \frac{1}{6}.$$

   (b)

$$\frac{\frac{\binom{6}{4}}{\binom{17}{4}}}{\sum_{i=1}^{6} \frac{\binom{6}{i}}{\binom{17}{i}}}.$$

3. (a) Possible values are -2, -1, 0, 1, 2, 4.

   (b) Set $q \stackrel{\text{def}}{=} 1/\binom{14}{2} = 1/91$. Then

$$p_X(j) = \begin{cases} \binom{4}{2}q & \text{if } j=4 \\ \binom{4}{1}\binom{2}{1}q & \text{if } j=2 \\ \binom{8}{1}\binom{4}{1}q & \text{if } j=1 \\ \binom{2}{2}q & \text{if } j=0 \\ \binom{8}{1}\binom{2}{1}q & \text{if } j=-1 \\ \binom{8}{2}q & \text{if } j=-2 \\ 0 & \text{else} \end{cases} = \begin{cases} 6q & \text{if } j=4 \\ 8q & \text{if } j=2 \\ 32q & \text{if } j=1 \\ q & \text{if } j=0 \\ 16q & \text{if } j=-1 \\ 28q & \text{if } j=-2 \\ 0 & \text{else} \end{cases}$$

4. (a)

$$p_X(j) = \begin{cases} \frac{2\binom{5}{2}}{\binom{10}{2}} & \text{if } j=2 \\ \frac{5\times 5}{\binom{10}{2}} & \text{if } j=-1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{20}{45} & \text{if } j=2 \\ \frac{25}{45} & \text{if } j=-1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{4}{9} & \text{if } j=2 \\ \frac{5}{9} & \text{if } j=-1 \\ 0 & \text{else} \end{cases}.$$

   (b)

$$\mathbb{E}[X] = \frac{2\times 4 - 1\times 5}{9} = \frac{3}{9} = \frac{1}{3}.$$

   (c)

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{4\times 4 + 1\times 5}{9} - \frac{1}{9} = \frac{20}{9}.$$

Math 461, Section F13, Fall 2008
Exam 3, November 14

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat

1. [43 points] Let the lifetime X of a lightbulb have hazard function

$$h(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t < 1 \\ \frac{3}{t} & \text{if } t \geq 1 \end{cases}$$

   (a) [15 points] Compute the cumulative distribution function of X (Hint: it is a simple expression).

   (b) [10 points] Compute the density of X.

   (c) [10 points] Compute $\mathbb{P}\{X > 2\}$

   (d) [8 points] Compute $\mathbb{E}[X]$.

2. [32 points] Let X be a Gaussian random variable with mean 0 and variance 4. Define $Y \stackrel{\text{def}}{=} X^2$.

   (a) [15 points] Compute the cumulative distribution function of Y.

   (b) [10 points] Compute the density of Y if it exists; if not, explain why the density doesn't exist.

   (c) [7 points] Compute $\mathbb{E}[Y]$. Note: this is a real easy question if you think about it.

3. [25 points] Let X be a Gaussian with mean 2 and variance 9. Use the attached table to compute

   (a) [10 points] $\mathbb{P}\{X > 4\}$

   (b) [15 points] $\mathbb{P}\{|X| \leq 5.5\}$

   Recall that the table gives the values of $\int_{-\infty}^{t}(2\pi)^{-1/2}e^{-x^2/2}dx$. In other words,

$$\int_{-\infty}^{2.14}(2\pi)^{-1/2}e^{-x^2/2}dx = .9838.$$

1. (a) Set $G_X(t) = \mathbb{P}\{X > t\} = 1 - F_X(t)$. Then

$$\dot{G}_X(t) = -f_X(t) = -h(t)G(t).$$

Also, $G_X(0) = 1 - F_X(0) = 1$, so $G_X(t) = \exp\left[-\int_{s=0}^t h(s)ds\right]$. Thus $G_X(t) = 1$ for $t < 1$. For $t > 1$,

$$G_x(t) = \exp\left[-\int_{s=1}^t \frac{3}{s}ds\right] = \exp\left[-3\ln t\right] = t^{-3}.$$

Thus

$$F_X(t) = 1 - G_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 - \frac{1}{t^3} & \text{if } t \geq 1 \end{cases}$$

(b)

$$f_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{3}{t^4} & \text{if } t \geq 1 \end{cases}$$

(c)

$$\mathbb{P}\{X > 2\} = 1 - F_X(2) = \frac{1}{2^3} = \frac{1}{8}$$

(d)

$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} t f_X(t)dt = 3\int_{t=1}^{\infty} \frac{t}{t^4}dt = 3\int_{t=1}^{\infty}\frac{1}{t^3}dt = \frac{3}{2}t^{-2}\Big|_{t=1}^{\infty} = \frac{3}{2}.$$

2. (a) We have that $F_Y(t) = \mathbb{P}\{Y \leq t\} = \mathbb{P}\{X^2 \leq t\}$. If $t < 0$, then $F_Y(t) = 0$. If $t \geq 0$, then

$$\begin{aligned} F_Y(t) = \mathbb{P}\{-\sqrt{t} \leq X \leq \sqrt{t}\} = F_X(\sqrt{t}) - F_X(-\sqrt{t}-) &= F_X(\sqrt{t}) - F_X(-\sqrt{t}) \\ &= \int_{z=-\sqrt{t}}^{\sqrt{t}} \frac{e^{-z^2/8}}{\sqrt{8\pi}}dz. \end{aligned}$$

(b) F_Y is continuous and differentiable (except at 0). If $t < 0$, then $f_Y(t) = 0$. If $t > 0$, then

$$f_Y(t) = \frac{1}{2\sqrt{t}}\left\{f_X(\sqrt{t}) + f_X(-\sqrt{t})\right\} = \frac{1}{\sqrt{8\pi t}}\exp\left[-\frac{t}{8}\right]$$

(c) $\mathbb{E}[Y] = \mathbb{E}[X^2] = 4$.

3. Let N be a standard normal. Then $X = 3N + 2$.

(a)

$$\begin{aligned} \mathbb{P}\{X > 4\} = \mathbb{P}\{3N + 2 > 4\} = \mathbb{P}\{N > 2/3\} = 1 - \mathbb{P}\{N < 2/3\} &= 1 - \Phi(.67) \\ &= 1 - .7486 = .2514 \end{aligned}$$

(b)

$$
\begin{aligned}
&\mathbb{P}\{|X| \le 5.5\} = \mathbb{P}\{-5.5 \le 3N+2 \le 5.5\} = \mathbb{P}\{-7.5 \le 3N \le 3.5\} \\
&= \mathbb{P}\left\{-\frac{15}{2} \le 3N \le \frac{7}{2}\right\} = \mathbb{P}\left\{-\frac{15}{6} \le N \le \frac{7}{6}\right\} = \mathbb{P}\left\{N \le \frac{7}{6}\right\} - \mathbb{P}\left\{N \le -\frac{15}{6}\right\} \\
&= \mathbb{P}\left\{N \le \frac{7}{6}\right\} - \left(1 - \mathbb{P}\left\{N > -\frac{15}{6}\right\}\right) = \mathbb{P}\left\{N \le \frac{7}{6}\right\} + \mathbb{P}\left\{N < \frac{15}{6}\right\} - 1 \\
&= \Phi(1.17) + \Phi(2.50) - 1 = .8790 + .9938 - 1 = .8728.
\end{aligned}
$$

Math 461, Section F13, Fall 2008
Final, December 16

Show all work; partial credit will be given
No calculators allowed
Remember that I have to grade this; don't over-reduce your answers
Remember that I have to grade this; be neat
200 points total

1. [32 points] This is similar to Quiz 1. Suppose that we have 30 people in a room, one of which is named Calvin and one of which is named Hobbes. We want to make two disjoint committees, where committee A has 5 people and committee B has 6 people.
   (a) [5 points] How many ways can these committees be formed?
   (b) [10 points] What is the probability that Calvin will be on committee A?
   (c) [10 points] What is the probability that Calvin is on committee A and Hobbes is on committee B?
   (d) [7 points] Given that Calvin is in Committee A, what is the probability that Hobbes will be in Committee B?

2. [15 points] This is an extension of quiz 3. At a certain college, 54% of the students are female. We also know that 10% of the students are majoring in computer science, and we know that 7% of the students are women majoring in computer science.
   (a) [10 points] Randomly select a computer science student. What is the probability that it is a woman?
   (b) [5 points] In this data, is gender independent of the decision to study computer science?

3. [22 points] Suppose that a random variable X has moment generating function

$$\mathbb{E}\left[e^{\theta X}\right] = \phi_X(\theta) = \begin{cases} \frac{2^3}{(2-\theta)^3} & \text{if } \theta < 2 \\ \infty & \text{else} \end{cases}$$

   (a) [10 points] Compute $\mathbb{E}[X]$
   (b) [7 points] Compute $\mathbb{E}[X^2]$
   (c) [5 points] Compute the variance of X.

4. [68 points] Suppose that we start are producing lightbulbs on an assembly line. Each light bulb is indepedent and it either works with probability p or doesn't work (with probability $1-p$). Let X be the position of the 3rd faulty lightbulb and let Y be the position of the 7th faulty lightbulb.
   (a) [10 points] Compute $\mathbb{P}\{X = 9\}$
   (b) [9 points] Compute $\mathbb{P}\{Y = 20\}$.

(c) $\boxed{10 \text{ points}}$ Compute $\mathbb{P}\{X = 9, Y = 20\}$.

(d) $\boxed{5 \text{ points}}$ Are X and Y independent?

(e) $\boxed{6 \text{ points}}$ Compute $\mathbb{P}\{X = 9|Y = 20\}$.

(f) $\boxed{7 \text{ points}}$ Compute the probability mass function $p_X(j)$.

(g) $\boxed{7 \text{ points}}$ Compute the probability mass function $p_Y(k)$.

(h) $\boxed{7 \text{ points}}$ Compute the joint probability mass function

$$p_{X,Y}(j,k) \stackrel{\text{def}}{=} \mathbb{P}\{X = j,\, Y = k\}.$$

(i) $\boxed{7 \text{ points}}$ Compute $\mathbb{P}\{X = j|Y = 20\}$.

Hint: Think about what it means that $X = 9$ and $Y = 20$.

5. $\boxed{17 \text{ points}}$ This is similar to question 2 on exam 3. Let X be a standard Gaussian (with mean 0 and variance 1). Define $Z \stackrel{\text{def}}{=} \ln|X|$.

(a) $\boxed{7 \text{ points}}$ Compute $\mathbb{P}\{Z \le 3\}$

(b) $\boxed{4 \text{ points}}$ Compute the cumulative distribution function F_Z of Z

(c) $\boxed{6 \text{ points}}$ Compute the density f_Z of Z if it exists. If it doesn't, state so.

6. $\boxed{46 \text{ points}}$ Let X and Y be independent random variables with density

$$f_X(t) = \begin{cases} 25te^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) = \begin{cases} 5e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{10 \text{ points}}$ Compute $\mathbb{E}\left[\frac{1}{X}\right]$.

(b) $\boxed{5 \text{ points}}$ Compute the joint density $f_{X,Y}(s,t)$ of X and Y.

(c) $\boxed{13 \text{ points}}$ Define $Z \stackrel{\text{def}}{=} X + Y$. Compute the density $f_Z(t)$ of Z.

(d) $\boxed{10 \text{ points}}$ Compute $\mathbb{P}\{X + 2Y \le 8\}$. Do as much of the integral as possible without getting really messy (use your judgement).

(e) $\boxed{8 \text{ points}}$ Define $U \stackrel{\text{def}}{=} X+2Y$. Compute the cumulative distribution F_U of U. Do as much of the integral as possible without getting really messy (use your judgement).

Answers

1. (a) $\binom{30}{5}\binom{25}{6}$

   (b)
$$\frac{\binom{29}{4}}{\binom{30}{5}} = \frac{(29)_4 5!}{4!(30)_5} = \frac{5}{30} = \frac{1}{6}.$$

   (c)
$$\frac{\binom{28}{4}\binom{24}{5}}{\binom{30}{5}\binom{25}{6}} = \frac{\frac{28!}{4!5!19!}}{\frac{30!}{5!6!19!}} = \frac{6 \times 5}{30 \times 29} = \frac{1}{29}$$

   (d)
$$\frac{\binom{28}{4}\binom{24}{5}\binom{30}{5}}{\binom{30}{5}\binom{25}{6}\binom{29}{4}} = \frac{6}{29}.$$

2.
$$W = \{\text{woman}\} \qquad \text{and} \qquad C = \{\text{computer science}\}.$$

   (a)
$$\mathbb{P}(W|C) = \frac{\mathbb{P}(W \cap C)}{\mathbb{P}(C)} = \frac{7}{10} = .7.$$

   (b) No; $\mathbb{P}(W|C) = .7 \neq .54 = \mathbb{P}(W)$.

3. (a) For $\theta < 2$, we have that $\phi'_X(\theta) = \frac{3\times 2^3}{(2-\theta)^4}$ so $\mathbb{E}[X] = \phi'_X(0) = \frac{3}{2}$.

   (b) For $\theta < 2$, we have that $\phi''_X(\theta) = \frac{12\times 2^3}{(2-\theta)^5}$, so $\mathbb{E}[X^2] = \frac{12}{4} = 3$.

   (c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{12-9}{4} = \frac{3}{4}$.

4. (a)
$$\begin{aligned}\mathbb{P}\{X = 9\} &= \mathbb{P}\{2 \text{ of the first 8 bulbs are faulty and the 9th bulb is faulty}\} \\ &= \binom{8}{2} p^6 (1-p)^3.\end{aligned}$$

   (b)
$$\begin{aligned}\mathbb{P}\{Y = 20\} &= \mathbb{P}\{6 \text{ of the first 19 bulbs are faulty and the 20th bulb is faulty}\} \\ &= \binom{19}{6} p^{13} (1-p)^7.\end{aligned}$$

   (c)
$$\begin{aligned}\mathbb{P}\{X = 9, Y = 20\} &= \mathbb{P}\{2 \text{ of the first 8 bulbs are faulty, the 9th bulb is faulty,} \\ &\qquad 3 \text{ of the next 10 bulbs are faulty, and the 20th bulb is faulty}\} \\ &= \binom{8}{2}\binom{10}{3} p^{13} (1-p)^7.\end{aligned}$$

(d) No; $\mathbb{P}\{X=9,Y=20\}\neq\mathbb{P}\{X=9\}\mathbb{P}\{Y=20\}$.

(e)

$$\mathbb{P}\{X=9|Y=20\}=\frac{\binom{8}{2}\binom{10}{3}}{\binom{19}{6}}$$

(f)

$$\begin{aligned}p_X(j)&=\mathbb{P}\{2\text{ of the first }j-1\text{ bulbs are faulty and the }j\text{-th bulb is faulty}\}\\&=\begin{cases}\binom{j-1}{2}p^{j-3}(1-p)^3 & \text{if } j\geq 3\\ 0 & \text{else}\end{cases}\end{aligned}$$

(g)

$$\begin{aligned}p_Y(k)&=\mathbb{P}\{6\text{ of the first }k-1\text{ bulbs are faulty and the }k\text{-th bulb is faulty}\}\\&=\begin{cases}\binom{k-1}{6}p^{k-7}(1-p)^7 & \text{if } k\geq 7\\ 0 & \text{else}\end{cases}\end{aligned}$$

(h)

$$\begin{aligned}p_{X,Y}(j,k)&=\mathbb{P}\{2\text{ of the first }j-1\text{ bulbs are faulty, the }j\text{-th bulb is faulty,}\\&\qquad 3\text{ of the next }k-1-j\text{ bulbs are faulty, and the }k\text{-th bulb is faulty}\}\\&=\begin{cases}\binom{j-1}{2}\binom{k-j-1}{3}p^{k-7}(1-p)^7 & \text{if } j\geq 3,\ k\geq j+4\\ 0 & \text{else}\end{cases}\end{aligned}$$

(i)

$$\mathbb{P}\{X=j|Y=20\}=\begin{cases}\frac{\binom{j-1}{2}\binom{19-j}{3}}{\binom{19}{6}} & \text{if } 3\leq j\leq 16\\ 0 & \text{else}\end{cases}$$

5. (a)

$$\mathbb{P}\{Z\leq 3\}=\mathbb{P}\{\ln|X|\leq 3\}=\mathbb{P}\left\{|X|\leq e^3\right\}=\mathbb{P}\{-e^3\leq X\leq e^3\}=\int_{s=-e^3}^{e^3}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds$$

(b)

$$F_Z(t)=\mathbb{P}\{\ln|X|\leq t\}=\mathbb{P}\left\{|X|\leq e^t\right\}=\mathbb{P}\left\{-e^t\leq |X|\leq e^t\right\}=\int_{s=-e^t}^{e^t}\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds.$$

(c) F_Z is continuous. We have that

$$f_Z(t)=\begin{cases}\frac{2}{\sqrt{2\pi}}\exp\left[t-\frac{1}{2}e^{2t}\right] & \text{if } t>0\\ 0 & \text{else}\end{cases}$$

6. (a)

$$\mathbb{E}\left[\frac{1}{X}\right] = \int_{s=-\infty}^{\infty} \frac{1}{t} f_X(t)dt = \int_{s=0}^{\infty} 25e^{-5t}dt = 5.$$

(b)

$$f_{X,Y}(s,t) = \begin{cases} 125se^{-5(s+t)} & \text{if } s \text{ and } t \text{ are positive} \\ 0 & \text{else} \end{cases}$$

(c) We can write that

$$f_X(t) = 25te^{-5t}\chi_{[0,\infty)}(t) \qquad \text{and} \qquad f_Y(t) = 5e^{-5t}\chi_{[0,\infty)}(t)$$

Note that $f_Y(t-s) = 5e^{-5(t-s)}\chi_{(-\infty,t]}(s)$. Thus

$$\begin{aligned} f_X(t) = \int_{s=-\infty}^{\infty} f_X(s)f_Y(t-s)ds &= \int_{s=-\infty}^{\infty} 25se^{-5s}\chi_{[0,\infty)}(s)5e^{-5(t-s)}\chi_{(-\infty,t]}(s)ds \\ &= 125e^{-5t}\int_{s=-\infty}^{\infty} s\chi_{[0,t]}(s)ds \\ &= \frac{125}{2}t^2e^{-5t}\chi_{[0,\infty)}(t) = \begin{cases} \frac{125}{2}t^2e^{-5t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

(d)

$$\mathbb{P}\{X + 2Y \le 8\} = \iint_{s+2t\le 8} f_{X,Y}(s,t)ds\,dt = \int_{t=0}^{4}\int_{s=0}^{8-2t} 125se^{-5(s+t)}ds\,dt.$$

(e) If $t < 0$, then $\mathbb{P}\{U \le t\} = 0$. If $t \ge 0$, then

$$\mathbb{P}\{X + 2Y \le t\} = \iint_{r+2s\le 8} f_{X,Y}(r,s)dr\,ds = \int_{s=0}^{t/2}\int_{r=0}^{t-2s} 125re^{-5(r+s)}dr\,ds.$$