Math 361, Section D13 and D14, Fall 2007
Quiz 1, August 24 **Name:** ____________________

1. 10 points Suppose that we have
   - 4 history books
   - 3 language books
   - 7 math books
   - 2 science books
   - 4 psychology books

   We want to arrange them on our bookshelf by subject. How many ways can we do this?

Answers

1.

$$5! \times 4! \times 3! \times 7! \times 2! \times 4!.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 2, August 29 **Name:** ____________________

1. This is question 7 on page 16 of the book.
   (a) [2 points] How many ways can we seat 3 boys and 3 girls in a row?
   (b) [2 points] How many ways can we seat 3 boys and 3 girls in a row if the boys and girls are each to sit together?
   (c) [3 points] How many ways can we seat 3 boys and 3 girls in a row if only the boys must sit together?
   (d) [3 points] How many ways can we seat 3 boys and 3 girls in a row if no two people of the same sex are allowed to sit together?

Answers

1. (a) $6!$
   (b) $2 \times 3! \times 3!$ (the 2 comes from which gender is left-most)
   (c) $4 \times 3! \times 3!$ (the 4 comes from the position of the left-most boy)
   (d) $2 \times 3! \times 3!$ (the two comes from which gender is left-most).

Math 361, Section D13 and D14, Fall 2007
Quiz 3, September 10 **Name:** ____________________

1. [10 points] This is question 10 on p. 112. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade, given that the second and third cards are spades.

Answers

1.
$$A = \{\text{first card is a spade}\} \qquad B = \{\text{second and third are spades}\}$$

Then

$$A \cap B = \{\text{all three cards are spades}\}$$

so

$$\mathbb{P}(A \cap B) = \frac{(13)_3}{(52)_3} \qquad \mathbb{P}(B) = \frac{(13)_2}{(52)_2}$$

and thus

$$\mathbb{P}(A|B) = \frac{(13)_3 \times (52)_2}{(13)_2 \times (52)_3} = \frac{11}{50}.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 4, September 24 Name: ____________________

1. 10 points This is question 62 on page 119. Barbara and Dianne shoot at a target. They are independent shooters. Suppose that Barbara hits the target with probability p_B and Dianne hits the target with probability p_D. Suppose that we know that the target was hit.

   (a) 5 points What is the probability that Barbara hit the target?

   (b) 5 points What is the probability that both Barbara and Dianne hit the target?

Answers

1. Let
$$B = \{\text{Barbara hits the target}\} \qquad D = \{\text{Dianne hits the target}\}.$$
The target being hit is the set $B \cup D$, and
$$\mathbb{P}(B \cup D) = \mathbb{P}(B) + \mathbb{P}(D) - \mathbb{P}(B \cap D) = p_B + p_D - p_B p_D.$$
   (a) Note that $B \subset (B \cup D)$, so
$$\mathbb{P}(B|B \cup D) = \frac{\mathbb{P}(B)}{\mathbb{P}(B \cup D)} = \frac{p_B}{p_B + p_D - p_B p_D}.$$
   (b) Note that $B \cap D \subset (B \cup D)$, so
$$\mathbb{P}(B \cap D|B \cup D) = \frac{\mathbb{P}(B \cap D)}{\mathbb{P}(B \cup D)} = \frac{p_B p_D}{p_B + p_D - p_B p_D}.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 5, October 1 **Name:** ____________________

1. [10 points] Consider an unfair die with

$$\mathbb{P}\{1\} = \mathbb{P}\{2\} = \mathbb{P}\{3\} = \mathbb{P}\{4\} = \frac{1}{6} \qquad \mathbb{P}\{5\} = \frac{1}{12} \qquad \text{and} \qquad \mathbb{P}\{6\} = \frac{3}{12}.$$

Assume also that X is a random variable defined as

$$X(1) = X(2) = 1, \qquad X(3) = X(4) = 2, \qquad \text{and} \qquad X(5) = X(6) = 3.$$

(a) [5 points] Compute $\mathbb{E}[X]$.

(b) [5 points] Compute $\mathbb{E}[X^2]$.

You don't need to reduce the answers.

Answers

1. (a)

$$\mathbb{E}[X] = \sum_{j=1}^{6} X(j)\mathbb{P}\{j\} = 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{12} + 3 \times \frac{3}{12}.$$

(b)

$$\mathbb{E}[X^2] = \sum_{j=1}^{6} X^2(j)\mathbb{P}\{j\} = 1 \times \frac{1}{6} + 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{12} + 9 \times \frac{3}{12}.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 6, October 8 **Name:** ____________________

1. 10 points This is a review of useful calculus. Prove that for any real number λ,

$$\lim_{n\to\infty}\left(1+\frac{\lambda}{n}\right)^n = e^\lambda.$$

If you use another result from calculus, clearly state what you are using.

1. We first note that by L'Hôpiltal's rule,

$$\lim_{x\nearrow\infty} x\ln\left(1+\frac{\lambda}{x}\right) = \lim_{x\nearrow\infty}\frac{\ln\left(1+\frac{\lambda}{x}\right)}{\frac{1}{x}} = \lim_{x\searrow 0}\frac{\ln(1+\lambda x)}{x} = \lim_{x\searrow 0}\frac{\lambda}{\ln(1+\lambda x)} = \lambda.$$

Since the exponential map is continuous,

$$\lim_{n\to\infty}\left(1+\frac{\lambda}{n}\right)^n = \exp\left[\lim_{x\nearrow\infty} x\ln\left(1+\frac{\lambda}{x}\right)\right] = e^\lambda.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 7, October 15 **Name:** ____________________

1. Let X be a random variable with probability mass function

$$p(j) = \begin{cases} \frac{2}{9} & \text{if } j = -1 \\ \frac{1}{9} & \text{if } j = 0 \\ \frac{2}{9} & \text{if } j = 3 \\ \frac{4}{9} & \text{if } j = 7 \\ 0 & \text{else} \end{cases}$$

Compute $\mathbb{E}[\min\{X, 1\}]$.

1.

$$
\begin{aligned}
\mathbb{E}[\min\{X,1\}] &= \sum_j \min\{j,1\}p(j) \\
&= \min\{-1,1\}\frac{2}{9} + \min\{0,1\}\frac{1}{9} + \min\{3,1\}\frac{2}{9} + \min\{7,1\}\frac{4}{9} \\
&= (-1)\frac{2}{9} + (0)\frac{1}{9} + (1)\frac{2}{9} + (1)\frac{4}{9} = \frac{-2+0+2+4}{9} = \frac{4}{9}.
\end{aligned}
$$

Math 361, Section D13 and D14, Fall 2007
Quiz 8, October 29 Name: ______________________

1. Let X be a continuous random variable with density

$$f(x) = \begin{cases} Cx^3 & \text{if } 0 \le x \le 2 \\ 0 & \text{else} \end{cases}$$

   (a) $\boxed{\text{5 points}}$ Find the constant C.

   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X^2]$.

1. (a) We must have

$$1 = \int_{x=-\infty}^{\infty} f(x)dx = C\int_{x=0}^{2} x^3 dx = C\frac{2^4}{4} = 4C$$

so $C = 1/4$.

(b) We compute that

$$\mathbb{E}[X^2] = \int_{x=-\infty}^{\infty} x^2 f(x)dx = \frac{1}{4}\int_{x=0}^{2} x^2 x^2 dx = \frac{1}{4}\frac{2^5}{5} = \frac{32}{4\times 5} = \frac{8}{5}.$$

Math 361, Section D13 and D14, Fall 2007
Quiz 9, November 5 **Name:** ____________________

1. 10 points Let X be a random variable with moment generating function

$$M(\theta) \stackrel{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \exp\left[\frac{9}{2}\theta^2\right]. \qquad \theta \in \mathbb{R}$$

Define $Y \stackrel{\text{def}}{=} 2X + 7$. Compute the moment generating function of Y.

Answers

1.

$$
\begin{aligned}
\mathbb{E}\left[\exp\left[\theta Y\right]\right] = \mathbb{E}\left[\exp\left[\theta(2X+7)\right]\right] = \mathbb{E}\left[\exp\left[(2\theta)X + 7\theta\right]\right] \\
= \mathbb{E}\left[\exp\left[(2\theta)X\right] e^{7\theta}\right] = e^{7\theta}\mathbb{E}\left[\exp\left[(2\theta)X\right]\right] = e^{7\theta}\exp\left[\frac{9}{2}(2\theta)^2\right] \\
= \exp\left[18\theta^2 + 7\theta\right].
\end{aligned}
$$

Math 361, Section D13 and D14, Fall 2007
Quiz 10, November 26 Name: ______________________

1. Let X and Y continuous random variables with joint density function

$$f_{X,Y}(s,t) = \begin{cases} e^{-t} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

   (a) 2 points Graph the area where $f_{X,Y} \neq 0$.

   (b) 4 points Find f_X

   (c) 4 points Find f_Y

Answers

1. (a) $f_{X,Y}(s,t)$ is nonzero above the line $s=t$ and in the first quadrant.

   (b)
$$f_X(s)=\int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt=\begin{cases}\int_{t=s}^{\infty} e^{-t}dt & \text{if } s>0\\ 0 & \text{else}\end{cases}=\begin{cases}e^{-t} & \text{if } s>0\\ 0 & \text{else}\end{cases}$$

   (c)
$$f_Y(t)=\int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds=\begin{cases}\int_{s=0}^{t} e^{-t}ds & \text{if } t>0\\ 0 & \text{else}\end{cases}=\begin{cases}te^{-t} & \text{if } t>0\\ 0 & \text{else}\end{cases}$$

Math 361, Section D13 and D14, Fall 2007
Quiz 11, December 3 **Name:** ______________________

1. Let X and Y be independent continuous random variables. Suppose that X is uniform $(0, 10)$ and Y is uniform $(0, 5)$. In other words,

$$f_X(t) = \begin{cases} \frac{1}{10} & \text{if } 0 < t < 10 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) = \begin{cases} \frac{1}{5} & \text{if } 0 < t < 5 \\ 0 & \text{else} \end{cases}$$

Define $Z \overset{\text{def}}{=} X + Y$, and let f_Z be the density of Z.

(a) [3 points] Compute $\mathbb{E}[X]$.

(b) [7 points] Compute f_Z.

1. (a) $\mathbb{E}[X] = \int_{s=-\infty}^{\infty} s f_X(s) ds = \frac{1}{10}\int_{s=0}^{10} s ds = 5.$

   (b) We can write

$$f_Z(t) = \frac{1}{10}\chi_{(0,10)}(t) \qquad \text{and} \qquad f_Y(t) = \frac{1}{5}\chi_{(0,5)}(t)$$

for all $t \in \mathbb{R}$. Then

$$\begin{aligned} f_Z(t) &= \int_{s=-\infty}^{\infty} f_X(s) f_Y(t-s) ds = \frac{1}{50}\int_{s=-\infty}^{\infty} \chi_{(0,10)}(s)\chi_{(0,5)}(t-s) ds \\ &= \frac{1}{50}\int_{s=-\infty}^{\infty} \chi_{(0,10)}(s)\chi_{(t-5,t)}(s) ds = \begin{cases} \frac{1}{50}\int_{s=0}^{t} ds & \text{if } t-5<0<t \\ \frac{1}{50}\int_{s=t-5}^{t} ds & \text{if } 0<t-5<10 \\ \frac{1}{50}\int_{s=t-5}^{10} ds & \text{if } t-5<10<t \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \frac{t}{50}\int_{s=0}^{t} ds & \text{if } 0<t<5 \\ \frac{5}{50}\int_{s=t-5}^{t} ds & \text{if } 5<t<10 \\ \frac{10-(t-5)}{50} & \text{if } 10<t<15 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{t}{50} & \text{if } 0<t<5 \\ \frac{5}{50} & \text{if } 5<t<10 \\ \frac{15-t}{50} & \text{if } 10<t<15 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Math 361, Section D13 and D14, Fall 2007
Quiz 12, December 7 **Name:** ____________________

1. Suppose that $\mathbb{P}(A) = 0.3$. Compute $\mathbb{P}(A^c)$.

Answers

1.
$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 1 - 0.3 = 0.7.$$

Math 361, Section D13 and D14, Fall 2007
Exam 1, September 17 Name: ____________

1. [50 points] This is question 52 on page 60. It is similar to the hat-matching problem.

   A box contains 10 pairs of shoes (we can label them as pairs 1 through 10). We grab 8 shoes and put them into a bag.

   (a) [10 points] What is the probability that the 1-st pair of shoes will be in the bag?

   (b) [10 points] What is the probability that the 4th and 7th pairs of shoes will be in the bag?

   (c) [10 points] What is the probability that the 2nd, 3rd, and 8th pairs of shoes will be in the bag?

   (d) [10 points] What is the probability that the 1st, 4th, 5th, and 10th pairs of shoes will be in the bag?

   (e) [10 points] What is the probability that there will be at least one (i.e., matching) pair of shoes in the bag? (Hint: use the inclusion-exlusion principle).

2. [15 points] This is essentially question 1 on page 111. Two fair dice are rolled.

   (a) [5 points] What is the probability that the two dice will show different numbers?

   (b) [10 points] What is the probability that at least one dice shows a 6 given that they show different numbers?

3. [15 points] This is essentially question 17 on page 112. In a certain community, $\frac{1}{3}$ of the people own a dog and $\frac{1}{5}$ own a cat. It also turns out that 10% of the dog-owners also own a cat.

   (a) [10 points] What is the probability that a randomly-selected family owns both a cat and a dog?

   (b) [5 points] What is the probability that a randomly-selected family which owns a dog given that it owns a cat?

4. [20 points] This question was motivated by problem 47 on page 60. There are 10 people in a room. Jane and Larry are among them.

   (a) [10 points] What is the probability that Jane and Larry were born in February and none of the other people in the room were born in February?

   (b) [10 points] What is the probability that exactly two of the people in the room were born in the same month?

Answers

1. (a)

$$\frac{\binom{2}{2}\binom{18}{6}}{\binom{20}{8}} = \frac{\binom{18}{6}}{\binom{20}{8}} = \frac{(8)_2}{(20)_2}.$$

(b)

$$\frac{\binom{4}{4}\binom{16}{4}}{\binom{20}{8}} = \frac{\binom{16}{4}}{\binom{20}{8}} = \frac{(8)_4}{(20)_4}.$$

(c)

$$\frac{\binom{6}{6}\binom{14}{2}}{\binom{20}{8}} = \frac{\binom{14}{2}}{\binom{20}{8}} = \frac{(8)_6}{(20)_6}.$$

(d)

$$\frac{\binom{8}{8}}{\binom{20}{8}} = \frac{1}{\binom{20}{8}} = \frac{8!}{(20)_6}.$$

(e)

$$\frac{10\binom{18}{6} - \binom{10}{2}\binom{16}{4} + \binom{10}{3}\binom{14}{2} - \binom{10}{4}}{\binom{20}{8}} = 1 - \frac{\binom{10}{8}2^8}{\binom{20}{8}}.$$

2. (a)

$$\frac{(6)_2}{36} = \frac{6\times 5}{36}.$$

(b)

$$\begin{aligned}&\mathbb{P}\{\text{at least one dice shows a 6 and they show different numbers}\}\\&\quad= \mathbb{P}\{\text{one dice shows a 6 and they show different numbers}\} = \frac{2\times 5}{36}.\end{aligned}$$

Thus the answer to the question is

$$\frac{2\times 5}{6\times 5} = \frac{2}{5}.$$

3. We have that

$$\mathbb{P}(D) = \frac{1}{3}, \qquad \mathbb{P}(C) = \frac{1}{5}, \qquad \text{and} \qquad \mathbb{P}(C|D) = \frac{1}{10}.$$

(a)

$$\mathbb{P}(C\cap D) = \mathbb{P}(C|D)\mathbb{P}(D) = \frac{1}{10}\times\frac{1}{3} = \frac{1}{30}.$$

(b)

$$\mathbb{P}(D|C) = \frac{\mathbb{P}(D\cap C)}{\mathbb{P}(C)} = \frac{\frac{1}{30}}{\frac{1}{5}} = \frac{5}{30} = \frac{1}{6}.$$

4. (a)

$$\frac{11^8}{12^{10}}.$$

(b)

$$12\binom{10}{2}\frac{(11)_8}{12^{10}}.$$

Math 361, Section D13 and D14, Fall 2007
Exam 2, October 19

1. [30 points] Suppose that X is a random variable with moment generating function

$$M(\theta) \stackrel{\text{def}}{=} \mathbb{E}[\exp[\theta X]] = \exp\left[5(e^{2\theta}-1)+3\theta\right]. \qquad \theta \in \mathbb{R}$$

   (a) [10 points] Compute $\mathbb{E}[X]$.

   (b) [10 points] Compute $\mathbb{E}[X^2]$.

   (c) [10 points] Compute the variance of X.

2. [20 points] Let X be geometric with parameter p; i.e.,

$$\mathbb{P}\{X=j\} = \begin{cases} (1-p)^{j-1}p & \text{for } j \in \{1,2,\dots\} \\ 0 & \text{else} \end{cases}$$

   Let's also define a new random variable $Y \stackrel{\text{def}}{=} \max\{X, 10\}$.

   (a) [10 points] Compute $\mathbb{E}[\max\{X,10\}]$. You don't need to compute the final summation.

   (b) [10 points] Compute $\mathbb{P}\{Y=10\}$. You do need to simplify as much as possible.

3. [20 points] This is close to question 13 on page 188. A salesman has two appointments. Each appointment will independently lead to a sale with probability p. Any sale made is equally likely to be either for \$500 or \$1,000. Let X be the total dollar amount of sales.

   (a) [10 points] Find the probability mass function of X

   (b) [10 points] Compute $\mathbb{E}[X]$.

4. [10 points] Suppose that X is a random variable with mean 20 and variance 5. Find a bounded interval (a,b) such that

$$\mathbb{P}\{X \notin (a,b)\} \leq \frac{5}{100}.$$

5. [20 points] This is essentially question 65 on page 195. Each of the 50,000 students in a university has a certain disease with probability 1/100. To prevent an epidemic, blood from all of the students is taken and combined and then the combined of blood is tested.

   (a) [10 points] What is the probability that the combined blood will test positive (i.e., at least one person has the disease)?

   (b) [10 points] The university should shut down if at least 50 people have the disease. Suppose that the combined blood tests positive. What is the (approximate) probability that the university needs to be shut down? You don't need to carry out the final numerical computation.

Answers

1. We first compute that

$$M'(\theta) = (10e^{2\theta}+3)M(\theta) \qquad \text{and} \qquad M''(\theta) = (20e^{2\theta})M(\theta) + (10e^{2\theta}+3)^2 M(\theta)$$

for all $\theta \in \mathbb{R}$.

(a) $\mathbb{E}[X] = M'(0) = 13$.

(b) $\mathbb{E}[X^2] = M''(0) = 20 + 13^2 = 189$.

(c) Variance of X is $20 + 13^2 - 13^2 = 20$.

2. (a)

$$\mathbb{E}[\max\{X,10\}] = \sum_{j=1}^{\infty} \max\{j,10\}(1-p)^{j-1}p = 10\sum_{j=1}^{10}(1-p)^{j-1}p + \sum_{j=1}^{10} j(1-p)^{j-1}p.$$

(b)

$$\mathbb{P}\{Y=10\} = \mathbb{P}\{X \le 10\} = \sum_{j=1}^{10}(1-p)^{j-1}p = p\sum_{j=0}^{9}(1-p)^j = p\frac{1-(1-p)^{10}}{1-(1-p)}$$
$$= 1-(1-p)^{10}.$$

3. Consider a single sale. The probability of making no sale is $1-p$. The probability of making a sale of \$500 is $p/2$. The probability of making a sale of \$1,000 is also $p/2$.

(a)

$$\mathbb{P}\{X=j\} = \begin{cases} (1-p)^2 & \text{for } j=0 \\ (1-p)p & \text{for } j=500 \\ p(1-p)+\frac{p^2}{4} & \text{for } j=1000 \\ \frac{p^2}{2} & \text{for } j=1500 \\ \frac{p^2}{4} & \text{for } j=2000 \end{cases}$$

(b)

$$\mathbb{E}[X] = 500p(1-p) + 1000p(1-p) + 250p^2 + 750p^2 + 1000p^2.$$

4. By Chebychev's inequality,

$$\mathbb{P}\{|X-20| \ge L\} \le \frac{5}{L^2}.$$

We want that $\frac{5}{L} \le \frac{5}{100}$; take $L=10$. Taking $a = 20-10 = 10$ and $b = 20+10 = 30$, we have that

$$\mathbb{P}\{X \notin (10,30)\} = \mathbb{P}\{|X-20| \ge 10\} \le \frac{5}{100} = 0.05.$$

5. This is a Poisson approximation problem. We let X be the number of people with the disease. The distribution of X will approximately be Poisson with parameter

$$50,000 \times \frac{1}{100} = 500.$$

(a) $\mathbb{P}\{X \geq 1\} = 1 - \mathbb{P}\{X = 0\} = 1 - e^{-500}$.

(b)

$$\mathbb{P}\{X \geq 50 | X \geq 1\} = \frac{\mathbb{P}\{X \geq 50\}}{\mathbb{P}\{X \geq 1\}} = (1 - e^{-500})^{-1} \sum_{j=50}^{\infty} e^{-500} \frac{(500)^j}{j!}.$$

Alternately, since there are only 50,000 students at the university, the sum in the numerator can have upper limit 50,000.

Math 361, Section D13 and D14, Fall 2007
Exam 3, November 12

1. 50 points Suppose that X is a continuous uniform$(0,1)$ random variable; i.e., it has density

$$f_X(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} X^3$.

(a) 10 points Compute $\mathbb{P}\left\{Y \le \frac{1}{5}\right\}$.

(b) 15 points Compute the cumulative distribution F_Y of Y

(c) 15 points Compute the density f_Y of Y, if it exists. If the density does not exist, explain why.

(d) 10 points Compute $\mathbb{E}[Y]$.

2. 20 points This is essentially question 21 on page 249 of the book. We will model the weight (in pounds) of 25-year old men as normal with mean $\mu = 170$ and variance $\sigma^2 = 25$. You should use the attached table of values of Gaussian integrals. Go as far as possible without doing messy math by hand.

(a) 10 points What percentage of 25-year old men weigh more than 165 pounds?

(b) 10 points What percentage of 25-year old men over 165 pounds are less than 180 pounds?

3. 30 points This is essentially question 7 on page 248 of the book. The density of a continuous random variable X is given by

$$f_X(t) = \begin{cases} a + bx^3 & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

If $\mathbb{E}[X] = \frac{3}{5}$, compute a and b.

Answers

1. (a)

$$\mathbb{P}\left\{Y \le \tfrac{1}{5}\right\} = \mathbb{P}\left\{X^3 \le \tfrac{1}{5}\right\} = \mathbb{P}\left\{X \le \left(\tfrac{1}{5}\right)^{1/3}\right\} = \int_{t=-\infty}^{(1/5)^{1/3}} f_Y(t)dt = (1/5)^{1/3}.$$

(b)

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \mathbb{P}\left\{X^3 \le t\right\} = \mathbb{P}\left\{X \le t^{1/3}\right\} = \int_{s=-\infty}^{t^{1/3}} f_Y(s)ds$$
$$= \begin{cases} 0 & \text{if } t^{1/3} < 0 \\ t^{1/3} & \text{if } 0 \le t^{1/3} < 1 \\ 1 & \text{if } t^{1/3} \ge 1 \end{cases} = \begin{cases} 0 & \text{if } t < 0 \\ t^{1/3} & \text{if } 0 \le t < 1 \\ 1 & \text{if } t \ge 1 \end{cases}$$

(c)

$$f_Y(t) = F_Y'(t) = \begin{cases} \frac{1}{3}t^{-2/3} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(d)

$$\mathbb{E}[Y] = \mathbb{E}[X^3] = \int_{t=-\infty}^{\infty} t^3 f_X(t)dt = \int_{t=0}^{1} t^3 dt = \frac{1}{4}$$

or equivalently

$$\mathbb{E}[Y] = \int_{t=-\infty}^{\infty} t f_Y(t)dt = \int_{t=0}^{1} t\left(\frac{1}{3}t^{-2/3}\right)dt = \frac{1}{3}\int_{t=0}^{1} t^{1/3}dt = \frac{1}{4}$$

2. Let ξ be normal with mean 0 and variance 1.

(a)

$$\mathbb{P}\{X \ge 165\} = \mathbb{P}\{5\xi + 170 \ge 165\} = \mathbb{P}\{\xi \ge -1\} = \mathbb{P}\{-\xi \ge -1\}$$
$$= \mathbb{P}\{\xi \le 1\} = 0.8413$$

(b)

$$\mathbb{P}\{X \le 180 | X \ge 165\} = \frac{\mathbb{P}\{165 \le X \le 180\}}{\mathbb{P}\{X \ge 165\}}$$

We calculate that

$$\mathbb{P}\{165 \le X \le 180\} = \mathbb{P}\{X \le 180\} - \mathbb{P}\{X < 165\}$$

We then compute that

$$\mathbb{P}\{X \le 180\} = \mathbb{P}\{5\xi + 170 \le 180\} = \mathbb{P}\{\xi \le 2\} = 0.9772.$$

We also have that

$$\mathbb{P}\{X < 165\} = 1 - \mathbb{P}\{X \geq 165\} = 1 - 0.8413.$$

Thus the final answer is

$$\frac{0.9772 - 1 + 0.8413}{0.8413} = \frac{0.8185}{0.8413}.$$

3. We need that

$$\int_{t=-\infty}^{\infty} f_X(t)dt = 1 \qquad \text{and} \qquad \int_{t=-\infty}^{\infty} t f_X(t)dt = \frac{3}{5}.$$

In other words,

$$\begin{aligned} 1 &= \int_{x=0}^{1} (a + bx^3)dx = a + \frac{b}{4} \\ \frac{3}{5} &= \int_{x=0}^{1} x(a + bx^3)dx = \frac{a}{2} + \frac{b}{5}. \end{aligned}$$

This gives us the two equations

$$4 = 4a + b \qquad \text{and} \qquad 6 = 5a + 2b.$$

The first equation is equivalent to $8 = 8a + 2b$, so we can subtract the second equation from this to get that $2 = 3a$; i.e., $a = \frac{2}{3}$. This then implies that $b = 4 - 4a = \frac{4}{3}$.

Math 361, Section D13 and D14, Fall 2007
Final, December 12

1. 34 points Suppose that we have 20 people in a math department, which includes Professor Integrate and Professor Differentiate. We need a four-person committee to govern the department. One of the committee members will be the chair.
   (a) 7 points How many ways are there to form the committee?
   (b) 7 points How many ways are there to form the committee and select the chair?
   (c) 10 points What is the probability that Professor Integrate will be on the committee?
   (d) 10 points What is the probability that Professor Differentiate will be the chair of the committee and Professor Integrate will not be on the committee?

2. 40 points This is inspired by Question 74 on page 121. Suppose that Fred and Jane roll a pair dice, with Fred starting first. They stop when either Fred rolls a 9 or Jane rolls a 4.
   (a) 5 points What is the probability of the sum of two dice being 9?
   (b) 5 points What is the probability of the sum of two dice being 4?
   (c) 10 points What is the probability that Jane wins on *her* third toss?
   (d) 10 points What is the probability that Jane wins on *her* n-th toss?
   (e) 10 points What is the probability that Jane wins? Be as explicit as possible without getting into messy calculations.

3. 18 points (Poisson approximation) This is inspired by question 51 on page 194. The expected number of typographical errors on a page is 0.01.
   (a) 9 points What is the probability that the first page has 2 or less typographical errors?
   (b) 9 points Assume that the first chapter has 10 pages. What is the probability that there are no typographical errors in the first chapter?

4. 58 points Suppose that X and Y are continuous random variables with joint density function

$$f_{X,Y}(s,t) = \begin{cases} e^{-t} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

   Define $Z \overset{\text{def}}{=} X + Y$.
   (a) 2 points Graph the region where $f_{X,Y}$ is positive
   (b) 10 points Compute f_X.
   (c) 10 points Compute f_Y.
   (d) 3 points Are X and Y independent?

(e) 10 points Compute $f_{X|Y}(s|t)$ for $t > 0$.

(f) 3 points Verbally describe the distribution of X if we know that $Y = 3$.

(g) 10 points Compute $f_{Y|X}(t|s)$ for $s > 0$.

(h) 10 points Compute f_Z (be aware of your answer to part d).

Answers

1. (a) $\binom{20}{4}$.

   (b) $\binom{20}{4} \times 4 = 20 \times \binom{19}{3}$.

   (c)

$$\mathbb{P}\{\text{Integrate is on committee}\} = \frac{1 \times \binom{19}{3}}{\binom{20}{3}} = \frac{4}{20} = \frac{1}{5}.$$

$$\begin{aligned}\mathbb{P}\{\text{Differentiate is chair and Integrate is not on committee}\} &= \frac{1 \times \binom{13}{3}}{20 \times \binom{19}{3}} \\ &= \frac{1}{20}\frac{(18)_3}{(19)_3}.\end{aligned}$$

2. (a)

$$\mathbb{P}\{9\} = \mathbb{P}\{(3,6),(4,5),(5,4),(6,3)\} = \frac{4}{36}.$$

   (b)

$$\mathbb{P}\{4\} = \mathbb{P}\{(1,3),(2,2),(3,1)\} = \frac{3}{36}.$$

   (c)

$$\mathbb{P}\{\text{Jane wins on her third toss}\} = \left(\frac{32}{36}\right)^3 \left(\frac{33}{36}\right)^2 \frac{3}{36}.$$

   (d)

$$\mathbb{P}\{\text{Jane wins on her n-th toss}\} = \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36}.$$

   (e)

$$\begin{aligned}\mathbb{P}\{\text{Jane wins}\} &= \sum_{n=1}^{\infty} \left(\frac{32}{36}\right)^{n-1} \left(\frac{33}{36}\right)^n \frac{3}{36}\frac{33}{36} \times \frac{3}{36}\sum_{n=1}^{\infty}\left(\frac{32}{36} \times \frac{33}{36}\right)^{n-1} \\ &= \frac{33}{36} \times \frac{3}{36}\sum_{n=0}^{\infty}\left(\frac{32}{36} \times \frac{33}{36}\right)^n = \frac{\frac{33}{36} \times \frac{3}{36}}{1 - \frac{32}{36} \times \frac{33}{36}} \\ &= \frac{33 \times 3}{36^2 - 32 \times 33}\end{aligned}$$

3. If X is the number of errors (on one page), the Poisson approximation is that

$$\mathbb{P}\{X = k\} = \begin{cases} e^{-0.01}\frac{(0.01)^k}{k!} & \text{for } k \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

(a)

$$\mathbb{P}\{X \le 2\} = 1 - \mathbb{P}\{X = 0\} + \mathbb{P}\{X = 1\} + \mathbb{P}\{X = 2\}$$
$$= e^{-0.01}\left\{1 + \frac{(0.01)}{1} + \frac{(0.01)^2}{2!}\right\} = 1.01005e^{-0.01}.$$

(b) For one page, $\mathbb{P}\{X = 0\} = e^{-0.01}$. Thus

$$\mathbb{P}\{\text{no errors in first chapter}\} = (e^{-0.01})^{10} = e^{-0.1}.$$

(a) First quadrant, above the line of 45 degrees.

(b)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^{\infty} e^{-t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

(c)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{t} e^{-t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} te^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

(d) No, since $f_{X,Y}(s,t) \neq f_X(s)f_Y(t)$.

(e) For $t > 0$,

$$f_{X|Y}(s|t) = \frac{f_{X,Y}(s,t)}{f_Y(t)} = \begin{cases} \frac{e^{-t}}{te^{-t}} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{t} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

(f) Uniform on $(0,3)$

(g) For $s > 0$,

$$f_{Y|X}(t|s) = \frac{f_{X,Y}(s,t)}{f_X(s)} = \begin{cases} \frac{e^{-t}}{e^{-s}} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-(t-s)} & \text{if } 0 \le s \le t \\ 0 & \text{else} \end{cases}$$

(h)

$$f_Z(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t-s)ds = \int_{s=-\infty}^{\infty} e^{-(t-s)}\chi_{[0,t-s]}(s)ds$$
$$= \begin{cases} \int_{s=0}^{t/2} e^{-(t-s)}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} \int_{u=t/2}^{t} e^{-u}du & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} e^{-t/2} - e^{-t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$