Math 361, Section A1, Summer 1996
Exam 1, June 25

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [15 points] Evaluate the following:
   (a) [5 points] $3!$
   (b) [5 points] $\binom{6}{4}$
   (c) [5 points] $\frac{6!}{4!}$

2. [40 points] Suppose we have two sets A and B. Suppose we know that

$$\mathbb{P}(A) = 0.3 \qquad \mathbb{P}(B^c) = 0.5 \qquad \text{and} \qquad \mathbb{P}(A \setminus B) = 0.2.$$

   (a) [5 points] $\mathbb{P}(A \cap B)$
   (b) [5 points] $\mathbb{P}(B)$
   (c) [5 points] $\mathbb{P}(B \setminus A)$
   (d) [5 points] $\mathbb{P}(A^c \cup B)$
   (e) [5 points] $\mathbb{P}(B^c | A)$
   (f) [5 points] $\mathbb{P}(A \cup B)$
   (g) [5 points] $\mathbb{P}(B^c | A \cup B)$
   (h) [5 points] Are the sets A and B independent? Why or why not?

3. [30 points] A janitor has 10 keys, exactly one of which fits a certain lock. She tries the keys one at a time, at each trial choosing at random from the keys that were not tried earlier.
   (a) [15 points] Find the probability that the 6th key tried is the correct key.
   (b) [15 points] Find the probability that the 6th key tried is the correct key, given that the first two keys don't work.

4. [15 points] A box contains 80 good fuses and 20 bad fuses. Select 5 fuses from the box. What is the probability that 4 of these are good and one is bad?

Answers

1. (a) 6
   (b) $\frac{6\cdot 5}{2} = 15$.
   (c) 30.

2. (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \setminus B) = 0.3 - 0.2 = 0.1$.
   (b) $\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - 0.5 = 0.5$.
   (c) $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B) = 1 - \mathbb{P}(B^c) - \mathbb{P}(A \cap B) = 1 - 0.5 - 0.1 = 0.4$.
   (d) $\mathbb{P}(A^c \cup B) = 1 - \mathbb{P}(A \setminus B) = 1 - 0.2 = 0.8$.
   (e) $\mathbb{P}(B^c|A) = \frac{\mathbb{P}(A\setminus B)}{\mathbb{P}(A)} = \frac{0.2}{0.3} = \frac{2}{3}$.
   (f) $\mathbb{P}(A \cup B) = \mathbb{P}(B) + \mathbb{P}(A \setminus B) = 1 - \mathbb{P}(B^c) + \mathbb{P}(A \setminus B) = 1 - 0.5 + 0.2 = 0.7$.
   (g) $\mathbb{P}(B^c|A \cup B) = \frac{\mathbb{P}(B^c\cap(A\cup B))}{\mathbb{P}(A\cup B)} = \frac{\mathbb{P}(A\setminus B)}{\mathbb{P}(A\cup B)} = \frac{0.2}{0.7} = \frac{2}{7}$.
   (h) No, since $\mathbb{P}(B^c|A) = \frac{2}{3} \neq 0.5 = \mathbb{P}(B^c)$.

3. $$A \overset{\text{def}}{=} \{\text{6th key works}\} \qquad B \overset{\text{def}}{=} \{\text{first two keys do not work}\}.$$
   (a)
   $$\mathbb{P}(A) = \frac{(9)_5 \cdot 1}{(10)_6} = \frac{1}{10}.$$
   (b) Note that $A \subset B$ and that $\mathbb{P}(B) = \frac{(9)_2}{(10)_2} = \frac{8}{10}$. Thus $\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = \frac{1}{8}$.

4. $$\frac{20\binom{60}{4}}{\binom{80}{5}}.$$

Math 361, Section A1, Summer 1996
Exam 2, July 11

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] A random variable X has moment generating function

$$\Phi_X(t) \stackrel{\text{def}}{=} \mathbb{E}\left[t^X\right] = \exp[\lambda(t^2 - 1)].$$

   (a) [5 points] Find $\mathbb{E}[X]$.

   (b) [5 points] Find $\mathbb{E}[X^2]$.

   (c) [5 points] Find $\mathrm{Var}(X)$.

   (d) [5 points] Find $\mathbb{E}[X^3]$ (Hint: what is the third derivative of $\Phi_X(1)$?).

   `Note:  This is not a good problem`

2. [15 points] Suppose that X and Y are independent random variables such that

$$\mathbb{E}[X^2] = 4 \qquad \mathbb{E}[Y^2] = 2 \qquad \mathbb{E}[X^2] = 1 \qquad \text{and} \qquad \mathbb{E}[Y] = 0.$$

   Compute the variance of $Z \stackrel{\text{def}}{=} X^2 Y$.

3. [15 points] Suppose a box has 20 balls labelled $1, 2 \ldots 20$. Two balls are selected from the box (without replacement). Let X denote the larger of the two numbers on the balls.

   (a) [10 points] Compute the density of X.

   (b) [5 points] Compute $\mathbb{E}[X]$.

4. [50 points] Let X and Y be two random variables having the joint density given by the following table:

		Y: -1	Y: 0	Y: 2	Y: 6
X	-2	1/9	1/27	1/27	1/9
X	1	2/9	0	1/9	1/9
X	3	0	0	1/9	4/27

   (a) [2 points] Compute $\mathbb{P}\{XY \text{ is odd}\}$.

   (b) [2 points] Compute $\mathbb{P}\{X \geq 0 \text{ and } Y < 0\}$.

   (c) [5 points] Compute the density of X.

   (d) [5 points] Compute the density of Y.

   (e) [5 points] Compute $\mathbb{E}[X]$.

(f) $\boxed{\text{3 points}}$ Compute $\mathbb{E}[Y]$.

(g) $\boxed{\text{5 points}}$ Find $\mathbb{E}[X^2]$.

(h) $\boxed{\text{3 points}}$ Find $\mathbb{E}[Y^2]$.

(i) $\boxed{\text{5 points}}$ Find $\text{Var}(X)$.

(j) $\boxed{\text{5 points}}$ Find $\text{Var}(Y)$.

(k) $\boxed{\text{5 points}}$ Find $\text{Cov}(X, Y)$.

(l) $\boxed{\text{5 points}}$ Find $\rho(X, Y)$.

Answers

1. Note that $\Phi'_X(t) = 2\lambda t \Phi_X(t)$; thus

$$\begin{aligned}\Phi''_X(t) &= \{2\lambda + 4\lambda^2 t^2\}\Phi_X(t)\\ \Phi'''_X(t) &= \{8\lambda^2 t + (2\lambda + 4\lambda^2 t^2)(2\lambda t)\}\Phi_X(t)\end{aligned}$$

   (a) $\mathbb{E}[X] = \Phi'_X(1) = 2\lambda$.

   (b) $\mathbb{E}[X(X-1)] = \Phi''_X(1) = 2\lambda+4\lambda^2$, so $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)]+\mathbb{E}[X] = 2\lambda+4\lambda^2-2\lambda = 4\lambda^2$.

   (c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = 4\lambda^2 - 4\lambda^2 = 0$. `This indicates that the problem is poorly written.`

   (d) $\mathbb{E}[X(X-1)(X-2)] = \Phi'''_X(1) = 8\lambda^2 + 4\lambda^2 + 8\lambda^3 = 12\lambda^2 + 8\lambda^3$, so

$$\begin{aligned}\mathbb{E}[X^3] &= \mathbb{E}[X(X-1)(X-2)] + 3\mathbb{E}[X^2] - 2\mathbb{E}[X]\\ &= 12\lambda^2 + 8\lambda^3 + 12\lambda^2 - 4\lambda = 8\lambda^3 + 24\lambda^2 - 4\lambda.\end{aligned}$$

2.

$$\mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \mathbb{E}[X^4Y^2] - \mathbb{E}[X^2Y]^2 = \mathbb{E}[X^4]\mathbb{E}[Y^2] - \mathbb{E}[X^2]^2\mathbb{E}[Y]^2 = 4\cdot 2 - 1^2\cdot 0 = 8.$$

3. (a) Note that $2 \le X \le 20$. For $j \in \{2,3\dots 20\}$,

$$\mathbb{P}\{X \le j\} = \frac{\binom{j}{2}}{\binom{20}{2}}.$$

   Thus, for $3 \le X \le 20$,

$$p_X(j) = \mathbb{P}\{X \le j\} - \mathbb{P}\{X \le j-1\} = \frac{\binom{j}{2} - \binom{j-1}{2}}{\binom{20}{2}} = \frac{j-1}{190}$$

   (check by direct computation that $\binom{j}{2} - \binom{j-1}{2} - j - 1$). We also have that

$$p_X(2) = \mathbb{P}\{X \le 2\} = \frac{1}{190}.$$

   For all other j, $p_X(j) = 0$.

   (b)

$$\mathbb{E}[X] = \sum_{j=2}^{20} j\frac{j-1}{190}.$$

4. (a) 2/9+0=2/9.

   (b) 2/9+0=2/9.

(c)

$$p_X(j) = \begin{cases} \frac{8}{27} & \text{if } j=-2 \\ \frac{12}{27} & \text{if } j=1 \\ \frac{7}{27} & \text{if } j=3 \\ 0 & \text{else} \end{cases}$$

(d)

$$p_Y(j) = \begin{cases} \frac{9}{27} & \text{if } j=-1 \\ \frac{1}{27} & \text{if } j=0 \\ \frac{7}{27} & \text{if } j=2 \\ \frac{10}{27} & \text{if } j=6 \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{E}[X] = \frac{1}{27}\{(-2)8 + (1)12 + 3(7)\} = \frac{1}{27}\{-16+12+21\} = \frac{17}{27}.$$

(f)

$$\mathbb{E}[Y] = \frac{1}{27}\{(-1)9 + (0)1 + (2)7 + (6)10\} = \frac{-9+14+60}{27} = \frac{65}{27}.$$

(g)

$$\mathbb{E}[X^2] = \frac{1}{27}\{(-2)^2 8 + (1)^2 12 + (3)^7 7\} = \frac{32+12+63}{27} = \frac{107}{27}.$$

(h)

$$\mathbb{E}[Y^2] = \frac{1}{27}\{(-1)^2 9 + (0)^2 1 + (2)^2 7 + (6)^2 10\} = \frac{9+28+360}{27} = \frac{397}{27}.$$

(i) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{107}{27} - \frac{289}{729} = \frac{2600}{729}$.

(j) $\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{397}{27} - \frac{4225}{729} = \frac{6494}{729}$.

(k)

$$\mathbb{E}[XY] = \frac{1}{27}\{(-2)(-1)3 + (1)(-1)6 + (-2)(2)1 + (1)(2)3$$
$$+(3)(2)3 + (-2)(6)3 + (1)(6)3 + (3)(6)4\} = \frac{6-6-4+6+18-36+18+72}{27}$$
$$= \frac{74}{27}.$$

Thus

$$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \frac{74}{27} - \left(\frac{17}{27}\right)\left(\frac{65}{27}\right) = \frac{74}{27} - \frac{1105}{729} = \frac{893}{27}.$$

(l)

$$\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} = \frac{893}{\sqrt{(2600)(6494)}}.$$

Math 361, Section A1, Summer 1996
Exam 3, July 24

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [38 points] Fix $a > 0$ and let X be a random variable uniformly distributed on $[0, a]$. Define a random variable Y by

$$Y = \begin{cases} a/3 & \text{if } X \le a/3 \\ X & \text{if } a/3 < X < 2a/3 \\ 2a/3 & \text{if } X \ge 2a/3. \end{cases}$$

   (a) [10 points] Find the distribution function of Y.
   (b) [4 points] What is $\mathbb{P}\{Y \le a/2\}$?
   (c) [5 points] What is $\mathbb{P}\{Y < a/2\}$?
   (d) [5 points] What is $\mathbb{P}\{Y = a/2\}$?
   (e) [4 points] What is $\mathbb{P}\{Y \le a/3\}$?
   (f) [5 points] What is $\mathbb{P}\{Y < a/3\}$?
   (g) [5 points] What is $\mathbb{P}\{Y = a/3\}$?

2. [32 points] Let X be a random variable with distribution function defined by

$$F_X(t) = \frac{1}{2} + \frac{t}{2(|t| + 1)}. \qquad t \in \mathbb{R}$$

   (a) [7 points] Express $\mathbb{P}\{|X - 1| \ge 2\}$ in terms of F.
   (b) [5 points] Does X have a density? If so, find it.
   (c) [10 points] Find the distribution function of the random variable $Y = |X|$.
   (d) [10 points] Does Y have a density? If so, find it.

3. [30 points] Let

$$f_{X,Y}(s,t) = \begin{cases} c(t-s)^\alpha & \text{if } 0 \le s < t \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) [10 points] For what values of α and c will f be a density function?
   (b) [15 points] For such a choice of α and c, what is f_X and f_Y?
   (c) [5 points] Are X and Y independent?

Answers

1. (a)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < a/3 \\ \frac{t}{a} & \text{if } a/3 \le t < 2a/3 \\ 1 & \text{if } t \ge 2a/3. \end{cases}$$

   (b) 1/2

   (c) 1/2

   (d) 0

   (e) 1/3

   (f) 0

   (g) 1/3.

2. (a)

$$\mathbb{P}\{|X-1| \ge 2\} = \mathbb{P}\{X \in (-\infty,-1] \cup [3,\infty)\} = F_X(-1) + 1 - F_X(3^-) = \frac{1}{4} + 1 - \frac{7}{8} = \frac{3}{8}.$$

   (b)

$$f_X(t) = \begin{cases} \frac{(t+1)-t}{2(t+1)^2} & \text{if } t > 0 \\ \frac{(1-t)+t}{2(1-t)} & \text{if } t < 0 \end{cases} = \begin{cases} \frac{1}{2(t+1)^2} & \text{if } t > 0 \\ \frac{1}{2(1-t)} & \text{if } t < 0 \end{cases} = \frac{1}{2(|t|+1)}.$$

   (c) $F_Y(t) = \mathbb{P}\{|X| \le t\}$. If $t < 0$, $F_Y(t) = 0$. If $t \ge 0$,

$$F_Y(t) = \mathbb{P}\{-t \le X \le t\} F_X(t) - F_X(-t^-) = \frac{t}{2(|t|+1)} - \frac{|-t|}{2(|-t|+1)} = \frac{t}{2(t+1)} + \frac{t}{2(t+1)} = \frac{t}{t+1}.$$

   (d)

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2(t+1)^2} & \text{if } t > 0 \end{cases}$$

3. (a)

$$1 = \int_{t\in\mathbb{R}} \int_{s\in\mathbb{R}} f_{X,Y}(s,t)\,ds\,dt = c\int_{t=0}^1 \int_{s=0}^t (t-s)^\alpha ds\,dt = \frac{c}{\alpha+1}\int_{t=0}^1 t^{\alpha+1} dt = \frac{c}{(\alpha+1)(\alpha+2)};$$

   for this to hold, we need that $\alpha > -1$; then $c = (\alpha+1)(\alpha+2)$.

(b)

$$f_X(s) = \int_{t\in\mathbb{R}} f_{X,Y}(s,t)dt = \begin{cases} c\int_{t=s}^1 (t-s)^\alpha dt & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{c(1-s)^{\alpha+1}}{\alpha+1} & \text{if } 0 < s < 1 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) = \int_{s\in\mathbb{R}} f_{X,Y}(s,t)ds = \begin{cases} c\int_{s=0}^t (t-s)^\alpha dt & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{ct^{\alpha+1}}{\alpha+1} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

(c) They are dependent, since if $0 < t < s < 1$, $f_{X,Y}(s,t) = 0 \neq f_X(s)f_Y(t)$.

Math 361, Section A1, Summer 1996
Final, August 3

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 200 Points

1. [10 points] Suppose that we have n balls and that we distribute them into r boxes. Let X be the number of balls which are in box 1. Find the density of X.

2. [10 points] Choose 4 cards from a standard deck of cards. Let X be the number of red cards which you have chosen. Find the density of X.

3. [20 points] Let X be geometrically distributed with parameter p.
   (a) [5 points] Compute $\mathbb{E}[X]$.
   (b) [5 points] Compute $\mathbb{E}[X^2]$.
   (c) [5 points] Compute the variance of X.
   (d) [5 points] Compute $\mathbb{E}[t^X]$.

4. [10 points] Let $X_1, X_2 \dots$ be random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 4^{-i}$. Compute
$$\mathbb{E}\left[\sum_{i=1}^{\infty} 2^i X_i^2\right].$$

5. [55 points] Let X and Y be independent geometrically distributed random variables with parameters p_1 and p_2 (respectively). Set $Z = X + Y$.
   (a) [10 points] Compute the density of Z.
   (b) [20 points] Compute the joint density of X and Z.
   (c) [10 points] Compute $p_{X|Z}(j|k)$, the conditional density of X given Z, where $k \geq 0$.
   (d) [15 points] Compute $\mathbb{E}[X|Z = k]$.

6. [50 points] Let X be a continuous random variable with density
$$f_X(t) = ce^{-|t|}. \qquad t \in \mathbb{R}$$
   Set $Y = X^3$.
   (a) [5 points] Compute what c must be.
   (b) [5 points] Compute $\mathbb{E}[X]$.
   (c) [10 points] Compute the cumulative distribution function of X.
   (d) [15 points] Compute the cumulative distribution function of Y.
   (e) [10 points] Compute the density of Y.

(f) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[Y]$.

7. $\boxed{\text{45 points}}$ Let

$$f(x,y) = \begin{cases} c(y-x)^\alpha & \text{if } 0 \le x < y \le 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) $\boxed{\text{10 points}}$ For what values of α and c will f be a density function?

(b) $\boxed{\text{20 points}}$ For such a choice of α and c, what is f_X and f_Y?

(c) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[X]$.

(d) $\boxed{\text{5 points}}$ Compute $\mathbb{E}[Y]$.

(e) $\boxed{\text{15 points}}$ Compute $\mathbb{E}[XY]$.

Answers

1. For $1 \le j \le n$,

$$p_X(j) = \frac{\binom{n}{j}(r-1)^{n-j}}{r^n}.$$

For all other j, $p_X(j) = 0$.

2. For $1 \le j \le n$,

$$p_X(j) = \frac{\binom{26}{j}\binom{26}{4-j}}{\binom{52}{4}}.$$

For all other j, $p_X(j) = 0$.

3.

$$\begin{aligned}\Phi_X(t) \stackrel{\text{def}}{=} \mathbb{E}\left[t^X\right] &= \sum_{k=1}^{\infty} p(1-p)^{k-1}t^j = pt\sum_{k=1}^{\infty}((1-p)t)^{k-1} \\ &= pt\sum_{k=0}^{\infty}((1-p)t)^k = \begin{cases}\frac{pt}{1-t(1-p)} & \text{if } t < 1/(1-p) \\ \infty & \text{else}\end{cases}\end{aligned}$$

For $t < 1/(1-p)$,

$$\Phi'_X(t) = \frac{p\{1-t(1-p)\} + (1-p)pt}{(1-t(1-p))^2} = \frac{p}{(1-t(1-p))^2}$$
$$\Phi''_X(t) = \frac{2p(1-p)}{(1-t(1-p))^3}.$$

(a) $\mathbb{E}[X] = \Phi'_X(1) = \frac{p}{p^2} = \frac{1}{p}$.

(b) $\mathbb{E}[X(X-1)] = \Phi''_X(1) = \frac{2p(1-p)}{p^3} = \frac{2(1-p)}{p^2}$, so $\mathbb{E}[X^2] = \mathbb{E}[X(X-1)] + \mathbb{E}[X] = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}$.

(c) $\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \frac{1-p}{p^2}$.

(d)

$$\mathbb{E}\left[t^X\right] = \begin{cases}\frac{pt}{1-t(1-p)} & \text{if } t < 1/(1-p) \\ \infty & \text{else}\end{cases}$$

4.

$$\mathbb{E}\left[\sum_{i=1}^{\infty} 2^i X_i^2\right] = \sum_{i=1}^{\infty} 2^i \mathbb{E}\left[X_i^2\right] = \sum_{i=1}^{\infty} 2^i 4^{-i} = \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$$

5. (a) For $j \in \{2, 3 \dots\}$,

$$p_Z(k) = \sum_{j=-\infty}^{\infty} \mathbb{P}\{X = j\}\mathbb{P}\{Y = k - j\} = p_1 p_2 \sum_{j=1}^{k-1} (1 - p_1)^{j-1}(1 - p_2)^{k-j-1}$$

$$= p_1 p_2 (1 - p_2)^{k-2} \sum_{j=1}^{k-1} \left(\frac{1 - p_1}{1 - p_2}\right)^{j-1} = p_1 p_2 (1 - p_2)^{k-2} \sum_{j=0}^{k-2} \left(\frac{1 - p_1}{1 - p_2}\right)^{j}$$

$$= p_1 p_2 (1 - p_2)^{k-2} \left\{ \frac{1 - \left(\frac{1-p_1}{1-p_2}\right)^{k-1}}{1 - \left(\frac{1-p_1}{1-p_2}\right)} \right\} = p_1 p_2 \left\{ \frac{(1 - p_2)^{k-1} - (1 - p_2)^{k-1}}{(1 - p_2) - (1 - p_1)} \right\}$$

$$= \frac{p_1 p_2}{p_1 - p_2} \left\{ (1 - p_2)^{k-1} - (1 - p_1)^{k-1} \right\}$$

if $p_1 \neq p_2$; if $p_1 = p_2 = q$, then

$$p_Z(k) = q^2 \sum_{j=1}^{k-1} (1 - q)^{j-1}(1 - q)^{k-j-1} = q^2 (1 - q)^{k-2} \sum_{j=1}^{k-1} 1 = q^2 (k - 1)(1 - q)^{k-2}.$$

(b)

$$p_{X,Z}(j, k) = \mathbb{P}\{X = j, Z = k\} = \mathbb{P}\{X = j, Y = k - j\}$$

$$= \begin{cases} p_1 p_2 (1 - p_1)^{j-1}(1 - p_2)^{k-j-1} & \text{if } j \geq 1 \text{ and } k - j \geq 1 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} p_1 p_2 (1 - p_1)^{j-1}(1 - p_2)^{k-j-1} & \text{if } 1 \leq j \leq k - 1 \\ 0 & \text{else} \end{cases}$$

(c)

$$p_{X|Z}(j|k) = \frac{p_{X,Z}(j, k)}{p_Z(k)}$$

If $1 \leq j \leq k - 1$,

$$p_{X|Z}(j|k) = \frac{(p_1 - p_2)(1 - p_1)^{j-1}(1 - p_2)^{k-j-1}}{(1 - p_2)^{k-1} - (1 - p_1)^{k-1}}$$

if $p_1 \neq p_2$, and

$$p_{X|Z}(j|k) = \frac{1}{k - 1}$$

if $p_1 = p_2 = q$.

(d) For $k \geq 0$,

$$\mathbb{E}[X|Z = k] = \sum_{j \in \mathbb{Z}} j p_{X|Z}(j|k).$$

Thus

$$\mathbb{E}[X|Z=k]=\sum_{j=1}^{k-1} j\frac{(p_1-p_2)(1-p_1)^{j-1}(1-p_2)^{k-j-1}}{(1-p_2)^{k-1}-(1-p_1)^{k-1}}$$

if $p_1 \neq p_2$, and

$$\mathbb{E}[X|Z=k]=\sum_{j=1}^{k-1}\frac{j}{k-1}=\frac{(k-1)k}{2(k-1)}=\frac{k}{2}$$

if $p_1 = p_2 = q$.

6.
$$1=\int_{t\in\mathbb{R}} f_X(t)dt = 2c\int_{t=0}^{\infty} = 2c;$$

$c=\frac{1}{2}$.

$$\mathbb{E}[X]=\frac{1}{2}\int_{t\in\mathbb{R}} te^{-|t|}dt=0$$

(note that f_X is even).

(b)

$$F_X(t)=\int_{s=-\infty}^{t} f_X(s)ds=\begin{cases}\frac{1}{2}\int_{s=-\infty}^{t} e^s ds & \text{if } t<0\\ \frac{1}{2}\int_{s=-\infty}^{0} e^s ds+\frac{1}{2}\int_{s=0}^{t} e^{-s}ds & \text{if } t\ge 0\end{cases}$$
$$=\begin{cases}\frac{e^t}{2} & \text{if } t<0\\ \frac{1}{2}+\frac{1}{2}\left(1-e^{-t}\right) & \text{if } t\ge 0\end{cases}=\begin{cases}\frac{e^t}{2} & \text{if } t<0\\ 1-\frac{e^{-t}}{2} & \text{if } t\ge 0\end{cases}$$

(c)

$$F_Y(t)=\mathbb{P}\{X^3\le t\}=F_X(t^{1/3})=\begin{cases}\frac{e^{t^{1/3}}}{2} & \text{if } t^{1/3}<0\\ 1-\frac{e^{-t^{1/3}}}{2} & \text{if } t^{1/3}\ge 0\end{cases}$$
$$=\begin{cases}\frac{e^{t^{1/3}}}{2} & \text{if } t<0\\ 1-\frac{e^{-t^{1/3}}}{2} & \text{if } t\ge 0\end{cases}$$

(d)

$$f_Y(t)=\begin{cases}\frac{|t|^{-2/3}}{6}e^{t^{1/3}} & \text{if } t<0\\ \frac{t^{-2/3}}{6}e^{-t^{1/3}} & \text{if } t\ge 0\end{cases}=\frac{|t|^{-2/3}}{6}e^{-|t|^{1/3}}$$

(e)

$$\mathbb{E}[Y]=\frac{1}{6}\int_{t\in\mathbb{R}} t|t|^{-2/3}e^{-|t|^{1/3}}dt=0$$

(note that f_Y is even).

7. (a)

$$1 = \int_{t\in\mathbb{R}}\int_{s\in\mathbb{R}} f_{X,Y}(s,t)ds\,dt = c\int_{t=0}^{1}\int_{s=0}^{t}(t-s)^\alpha ds\,dt$$
$$= \frac{c}{\alpha+1}\int_{t=0}^{1} t^{\alpha+1}dt = \frac{c}{(\alpha+1)(\alpha+2)};$$

for this to hold, we need that $\alpha > -1$; then $c = (\alpha+1)(\alpha+2)$.

(b)

$$f_X(s) = \int_{t\in\mathbb{R}} f_{X,Y}(s,t)dt = \begin{cases} c\int_{t=s}^{1}(t-s)^\alpha dt & \text{if } 0 < s < 1 \\ 0 & \text{else}\end{cases}$$
$$= \begin{cases} \frac{c(1-s)^{\alpha+1}}{\alpha+1} & \text{if } 0 < s < 1 \\ 0 & \text{else}\end{cases}$$
$$f_Y(t) = \int_{s\in\mathbb{R}} f_{X,Y}(s,t)ds = \begin{cases} c\int_{s=0}^{t}(t-s)^\alpha dt & \text{if } 0 < t < 1 \\ 0 & \text{else}\end{cases}$$
$$= \begin{cases} \frac{ct^{\alpha+1}}{\alpha+1} & \text{if } 0 < t < 1 \\ 0 & \text{else}\end{cases}$$

(c)

$$\mathbb{E}[X] = \int_{s\in\mathbb{R}} sf_X(s)ds = \frac{c}{\alpha+1}\int_{s=0}^{1} s(1-s)^{\alpha+1}ds.$$

(d)

$$\mathbb{E}[Y] = \int_{t\in\mathbb{R}} tf_Y(t)dt = \frac{c}{\alpha+1}\int_{t=0}^{1} t^{\alpha+2}ds = \frac{c}{(\alpha+1)(\alpha+2)} = \frac{\alpha+2}{\alpha+3}.$$

(e)

$$\mathbb{E}[XY] = \int_{s\in\mathbb{R}}\int_{t\in\mathbb{R}} stf_{X,Y}(s,t)ds\,dt = c\int_{t=0}^{1}\int_{s=0}^{t} st(t-s)^\alpha ds\,dt.$$