Math 361, Section E1, Spring 2003
Quiz 1, January 31 **Name:** ____________________

1. Suppose that

$$\mathbb{P}(A) = 0.3 \qquad \mathbb{P}(B) = 0.7 \qquad \text{and} \qquad \mathbb{P}(B \setminus A) = 0.5.$$

Compute $\mathbb{P}(A \setminus B)$.

1. First compute that $\mathbb{P}(A \cap B) = \mathbb{P}(B) - \mathbb{P}(B \setminus A) = 0.7 - 0.5 = 0.2$. Then we have that $\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(A \cap B) = 0.3 = 0.2 = 0.1$.

Math 361, Section E1, Spring 2003
Quiz 2, February 7 **Name:** ____________________

1. At a certain university, 53% of the students are women, and 5% of the students are math majors. Furthemore, 7% of the women are majoring in math. What is the probability that a randomly selected math student is a woman?

Answers

1. Define $W = \{\text{women}\}$ and $M = \{\text{math}\}$. Then we know that $\mathbb{P}(W) = 0.53$, $\mathbb{P}(M) = 0.05$, and $\mathbb{P}(M|W) = 0.07$. We then have that

$$\mathbb{P}(W|M) = \frac{\mathbb{P}(W \cap M)}{\mathbb{P}(M)} = \frac{\mathbb{P}(M|W)\mathbb{P}(W)}{\mathbb{P}(M)} = \frac{(0.07)(0.53)}{(0.05)}.$$

Math 361, Section E1, Spring 2003
Quiz 3, February 17 **Name:** ____________________

1. Rebecca and Veronica throw darts. Rebecca hits the target with probability 0.5 and Veronika hits the target with probability 0.7. Suppose that Rebecca and Veronica both independently throw a dart and that the target is hit. What is the probability that Veronica hits the target?

Answers

1. Let $R = \{\text{Rebecca hits the target}\}$ and $V = \{\text{Veronica hits the target}\}$. Then

$$\begin{aligned}\mathbb{P}(V|R\cup V) = \frac{\mathbb{P}(V)}{\mathbb{P}(R\cup V)} = \frac{\mathbb{P}(V)}{\mathbb{P}(R)+\mathbb{P}(V)-\mathbb{P}(R\cap V)} &= \frac{\mathbb{P}(V)}{\mathbb{P}(R)+\mathbb{P}(V)-\mathbb{P}(R)\mathbb{P}(V)}\\ &= \frac{0.7}{0.5+0.7-(0.5)(0.7)}.\end{aligned}$$

Math 361, Section E1, Spring 2003
Quiz 4, March 3 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that X is a geometric random variable with parameter p; i.e., it has probability mass function

$$p_X(i) = \begin{cases} p(1-p)^i & \text{if } i \in \{0,1\dots\} \\ 0 & \text{else} \end{cases}$$

Define now a new random variable

$$Y \stackrel{\text{def}}{=} \max\{X, 10\}.$$

(a) $\boxed{\text{5 points}}$ If $Y \geq 2$, what do we know about X?

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{Y \geq 2\}$.

Answers

1. (a) X can take on any value.

   (b) $\mathbb{P}\{Y \geq 2\} = 1$.

Math 361, Section E1, Spring 2003
Quiz 5, March 5 **Name:** ______________________

1. $\boxed{\text{10 points}}$ Suppose that X has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{3} & \text{if } 0 \le t < 1 \\ \frac{1}{2} & \text{if } 1 \le t < 2 \\ \frac{2}{3} + \frac{t-2}{3} & \text{if } 2 \le t < 3 \\ 1 & \text{if } t \ge 3 \end{cases}$$

(a) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X = 1\}$

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X > 2\}$.

Answers

1. (a) $\mathbb{P}\{X=1\} = F_X(1) - F_X(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.
   (b) $\mathbb{P}\{X>2\} = 1 - \mathbb{P}\{X \le 2\} = 1 - F_X(2) = 1 - \frac{2}{3} = \frac{1}{3}$.

Math 361, Section E1, Spring 2003

Quiz 6, March 17 **Name:** ____________________

1. Consider a binary communications channel. The probability of a single binary digit being correctly received is 0.8. Consider the following encoding scheme. If we wish to transmit a 0, we send the sequence 000. If we wish to transmit a 1, we send the sequence 111. The receiver uses "majority rule" decoding; i.e., if it receives 110, it decodes the sequence as 1. Suppose that we wish to transmit a 0. What is the probability that it will be decoded correctly?

Answers

1.
$$\mathbb{P}\{\text{either 2 or 3 zeroes are received}\} = (0.8)^3 + \binom{3}{2}(0.8)^2(0.2).$$

Math 361, Section E1, Spring 2003
Quiz 7, April 2 **Name:** ____________________

1. $\boxed{\text{10 points}}$ Suppose that X is a continuous random variable with density

$$f_X(t) = \begin{cases} 9e^{-9t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

   (a) $\boxed{\text{5 points}}$ Compute $\mathbb{E}\left[e^{2X}\right]$.

   (b) $\boxed{\text{5 points}}$ Compute $\mathbb{E}\left[e^{\theta X}\right]$ for all $\theta < 9$.

ANSWERS

1. (a)

$$\mathbb{E}\left[e^{6X}\right] = \int_{\mathbb{R}} e^{6t} f_X(t)dt = \int_0^\infty 9e^{-3t}dt = \frac{9}{3} = 3.$$

   (b) If $\theta < 9$,

$$\mathbb{E}\left[e^{\theta X}\right] = \int_{\mathbb{R}} e^{\theta t} f_X(t)dt = \int_0^\infty 9e^{-(9-\theta)t}dt = \frac{9}{9-\theta}.$$

Math 361, Section E1, Spring 2003
Quiz 8, April 9 **Name:** ______________________

1. Suppose that X is a continuous random variable with moment generating function

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\theta^2/2+5\theta} \qquad \theta \in \mathbb{R}$$

Define $Y \stackrel{\text{def}}{=} 7X$. Compute $\mathbb{E}\left[e^{\theta Y}\right]$ for all $\theta \in \mathbb{R}$.

1.

$$\mathbb{E}\left[e^{\theta Y}\right] = \mathbb{E}\left[\exp\left[\theta(7X)\right]\right] = \mathbb{E}\left[\exp\left[(7\theta)X\right]\right] = \exp\left[\frac{(7\theta)^2}{2} + 5(7\theta)\right]$$
$$= \exp\left[\frac{49}{2}\theta^2 + 35\theta\right]$$

for all $\theta \in \mathbb{R}$.

Math 361, Section E1, Spring 2003
Quiz 9, April 14 Name: ____________

1. $\boxed{\text{10 points}}$ Suppose that X is a continuous random variable which is uniform on $(0,1)$; i.e., it has density

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \frac{1}{X^2}$

(a) $\boxed{\text{3 points}}$ Compute $\mathbb{P}\{Y \le 9\}$.

(b) $\boxed{\text{2 points}}$ Compute the cumulative distribution function $F_Y(t)$ for $t \ge 1$.

(c) $\boxed{\text{5 points}}$ Compute the density $f_Y(t)$ for $t > 1$.

1. (a) $\mathbb{P}\{Y \le 9\} = \mathbb{P}\left\{|X| \ge \frac{1}{3}\right\} = \frac{2}{3}$.

   (b) $F_Y(t) = \mathbb{P}\left\{|X| \ge \frac{1}{\sqrt{t}}\right\} = 1 - \frac{1}{\sqrt{t}}$.

   (c) $f_Y(t) = \frac{1}{2}t^{-3/2}$.

Math 361, Section E1, Spring 2003
Quiz 10, April 16 Name: ____________________

1. $\boxed{\text{10 points}}$ Suppose that X is a continuous random variable which is exponentially distributed with parameter 2; i.e., it has density

$$f_X(t) = \begin{cases} 2e^{-2t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \max\{4.5, X\}$.

(a) $\boxed{\text{3 points}}$ Compute $\mathbb{P}\{Y \le 2.7\}$.

(b) $\boxed{\text{2 points}}$ Compute $\mathbb{P}\{Y \le 7\}$

(c) $\boxed{\text{5 points}}$ Compute the cumulative distribution function F_Y of Y.

1. (a) $\mathbb{P}\{Y \le 2.7\} = \mathbb{P}(\emptyset) = 0$.

   (b)
$$\mathbb{P}\{Y \le 7\} = \mathbb{P}\{X \le 7\} = \int_{s=-\infty}^{7} f_X(s)ds = \int_{s=0}^{7} 2e^{-2s}ds = 1 - e^{-14}.$$

   (c)
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 4.5 \\ 1 - e^{-2t} & \text{if } t \ge 4.5 \end{cases}$$

Math 361, Section E1, Spring 2003
Quiz 11, April 21 **Name:** ______________________

1. Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} 6t & \text{if } s \ge 0,\ t \ge 0, \text{ and } s+t \le 1 \\ 0 & \text{else} \end{cases}$$

Compute the density f_Y of Y.

1.

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{1-t} 6tds & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases} = \begin{cases} 6t(1-t) & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Spring 2003
Quiz 12, April 23 **Name:** ______________________

1. $\boxed{\text{10 points}}$ Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 3e^{-3(t-s)} & \text{if } t \ge s \text{ and } 0 \le s \le 1 \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{\text{5 points}}$ Compute $f_Y(6)$, where f_Y is the density of Y.

(b) $\boxed{\text{5 points}}$ Compute the conditional density $f_{X|Y}(s|6)$ for all $s \in \mathbb{R}$.

Answers

1. (a)

$$f_Y(6) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,6)ds = \int_{s=0}^{1} 3e^{-3(6-s)}ds = e^{-18}\left(e^3 - 1\right).$$

(b)

$$f_{X|Y}(s|6) = \frac{f_{X,Y}(s,6)}{f_Y(6)} = \begin{cases} \frac{3e^{-3(6-s)}}{e^{-18}(e^3-1)} & \text{if } s \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Spring 2003
Exam 1, February 21

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{40 points}}$ Suppose that we flip 5 independent coins. For each coin, $\mathbb{P}\{\text{heads}\} = p$ and $\mathbb{P}\{\text{tails}\} = 1 - p$.
   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{\text{HHHHT}\}$.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{\text{exactly one head}\}$.
   (c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{\text{HTTHH}\}$.
   (d) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{\text{exactly two heads}\}$.

2. $\boxed{\text{20 points}}$ A certain *system* consists of three *machines*, namely machines X, Y, and Z. Each machine is independent, and works with probability p (and fails with probability $1 - p$). The system works only if at least 2 machines work.
   (a) $\boxed{\text{10 points}}$ Compute the probability that the system is working.
   (b) $\boxed{\text{10 points}}$ Compute the probability that machine X is working given that the system is working.

3. $\boxed{\text{20 points}}$ Four cards are taken from a standard deck of cards. What is the probability that they are
   (a) $\boxed{\text{10 points}}$ of different face values.
   (b) $\boxed{\text{10 points}}$ of different suits.

4. $\boxed{\text{10 points}}$ Urn I contains 4 red and 4 black balls. Urn II contains 7 red and 11 black balls. A ball is selected from each urn. What is the probability that both balls are of the same color?

5. $\boxed{\text{10 points}}$ Two brothers are on the same team on a game show. Both brothers independently know the answer to any given question with probability p. They decide upon the following strategy to answer a question. If they agree on the answer, then that is their answer. If they disagree, they flip a fair coin. If the coin comes up tails, the younger brother answers the question, and if the coin comes up heads, the older brother answers the questions. What is the probability that their team answers a specific question right?

Answers

1. (a) $p^4(1-p)$.
   (b) $5p(1-p)^4$.
   (c) $p^3(1-p)^2$.
   (d) $\binom{5}{2}p^2(1-p)^3$.
2. (a) $p^3+3p^2(1-p)$.
   (b)
$$\frac{p^3+2p^2(1-p)}{p^3+3p^2(1-p)}.$$
3. (a)
$$\frac{\binom{13}{4}4^4}{\binom{52}{4}}.$$
   (b)
$$\frac{(13)^4}{\binom{52}{4}}.$$
4.
$$\frac{4}{8}\frac{7}{18}+\frac{4}{8}\frac{11}{18}.$$
5. $p^2+\frac{1}{2}p(1-p)+\frac{1}{2}p(1-p)=p^2+p(1-p)=p$.

Math 361, Section E1, Spring 2003
Exam 2, March 21

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{25 points}}$ Suppose that a random variable X has moment generating function

$$\varphi_X(\theta) = \mathbb{E}\left[e^{\theta X}\right] = \begin{cases} \frac{7}{7-\theta} & \text{if } \theta < 7 \\ \infty & \text{if } \theta \geq 7 \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X^2]$.

   (c) $\boxed{\text{5 points}}$ Compute the variance of X.

2. $\boxed{\text{30 points}}$ Suppose that we toss a sequence of biased coins ($\mathbb{P}\{\text{H}\} = p$). Let X be the position of the first heads. Compute

   (a) $\boxed{\text{10 points}}$ $\mathbb{P}\{\text{the second heads appears on the 10th toss}\}$.

   (b) $\boxed{\text{10 points}}$ $\mathbb{P}\{X = 7 \text{ and the second heads appears on the 10th toss}\}$.

   (c) $\boxed{\text{10 points}}$ $\mathbb{P}\{X = 7 | \text{the second heads appears on the 10th toss}\}$.

3. $\boxed{\text{10 points}}$ (This is essentially Question 5 on p. 228) Suppose that the demand for gasoline at a certain gas station is a continuous random variable with density

$$f_X(t) = \begin{cases} 2(1-t) & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

   Suppose that the owner wants to buy a new gasoline tank for the station. Find the capacity C of the tank so that the gas station will be sold out with probability 0.01.

4. $\boxed{\text{20 points}}$ (This is essentially question 20 in Chapter 4) Consider a roulette strategy. A roulette wheel can come up either red (R) or black (B). We have that]

$$\mathbb{P}\{\text{R}\} = \frac{18}{38} \qquad \text{and} \qquad \mathbb{P}\{\text{B}\} = \frac{20}{38}.$$

   On each game, we can bet \$1 on red. If it comes up red, we get our original dollar back and get one more dollar (winnings of \$1). If it comes up black, we lose our original dollar (winnings of $-$\$1). Consider the following strategy. Bet on red. If it comes up red, we quit. If it comes up black, we bet on red on the next two games (and then quit). Let X be our total winnings.

   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X = 1\}$.

   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$ (do not do the final computation).

5. $\boxed{\text{15 points}}$ Let X be a geometric random variable with parameter p; i.e., it has probability mass function

$$p_X(j) \overset{\text{def}}{=} \begin{cases} (1-p)^j p & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

(hint: you may want to remember that $\sum_{j=0}^{\infty} \alpha = (1-\alpha)^{-1}$ if $|\alpha| < 1$).

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X \geq 7\}$.

(b) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X \geq 7 | X \geq 3\}$.

Answers

1. Note that if $\theta < 2$,

$$\dot{\varphi}_X(\theta) = \frac{7}{(7-\theta)^2} \qquad \text{and} \qquad \ddot{\varphi}_X(t) = 2\frac{7}{(7-\theta)^3}$$

   (a) $\mathbb{E}[X] = \dot{\varphi}_X(0) = \frac{7}{7^2} = \frac{1}{7}$.
   (b) Compute $\mathbb{E}[X^2] = 2\frac{7}{7^3} = \frac{2}{49}$.
   (c) $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{49} - \frac{1}{29} = \frac{1}{49}$.

2. (a) $9p^2(1-p)^8$.
   (b) $p^2(1-p)^8$.
   (c)

$$\frac{p^2(1-p)^8}{9p^2(1-p)^8} = \frac{1}{9}.$$

3. We want $C \in (0,1)$ such that

$$0.01 = \int_C^1 f_X(t)dt = \int_C^1 2(1-t)dt = (1-C)^2$$

   so $C = 0.9$.

4. (a) $\mathbb{P}\{X = 1\} = \frac{18}{38} + \frac{20}{38}\left(\frac{18}{38}\right)^2$.
   (b)

$$\mathbb{E}[X] = (1)\frac{18}{38} + (-1)\cdot 2 \cdot \frac{18}{38}\left(\frac{20}{38}\right)^2 + (-3)\left(\frac{20}{38}\right)^3 + (1)\frac{20}{38}\left(\frac{18}{38}\right)^2.$$

5. (a) $\mathbb{P}\{X \geq 7\} = \sum_{j=7}^{\infty}(1-p)^j p = (1-p)^7$.
   (b)

$$\mathbb{P}\{X \geq 7 | X \geq 3\} = \frac{(1-p)^7}{(1-p)^3} = (1-p)^4.$$

Math 361, Spring 2003
Exam 2 (Makeup), March 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{40 points}}$ (Essentially Question 30 on p. 183). A jar contains 10 distinct chips. You successively draw chips from the jar, replacing the chip at each turn. Let X denote the number of draws until you select a chip which you had previously selected.
   (a) $\boxed{\text{6 points}}$ Verbally describe what happens when $X = 2$.
   (b) $\boxed{\text{6 points}}$ Compute $\mathbb{P}\{X = 2\}$.
   (c) $\boxed{\text{5 points}}$ Verbally describe what happens when $X = 3$.
   (d) $\boxed{\text{5 points}}$ Compute $\mathbb{P}\{X = 3\}$.
   (e) $\boxed{\text{4 points}}$ Verbally describe what happens when $X = 4$.
   (f) $\boxed{\text{4 points}}$ Compute $\mathbb{P}\{X = 4\}$.
   (g) $\boxed{\text{3 points}}$ Verbally describe what happens when $X = k$, for general $k \in \{2, 3 \dots\}$.
   (h) $\boxed{\text{7 points}}$ Compute $\mathbb{P}\{X = k\}$.

2. $\boxed{\text{20 points}}$ Suppose that X is a random variable with mean 27 and variance 2. Define also $Y \stackrel{\text{def}}{=} 2X$.
   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X^2]$.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[Y]$.

3. $\boxed{\text{20 points}}$ Suppose that X is a discrete random variable with probability mass function

$$p_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{12} & \text{if } j = 1 \\ \frac{3}{12} & \text{if } j = 2 \\ \frac{5}{12} & \text{if } j = 4 \\ \frac{3}{12} & \text{if } j = 10 \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.
   (b) $\boxed{\text{10 points}}$ Compute $\mathbb{E}\left[\frac{1}{X}\right]$.

4. $\boxed{\text{20 points}}$ Suppose that the lifetime of a certain computer component is a continuous random variable with density

$$f_X(t) = \begin{cases} 5e^{-5(t-7)} & \text{if } t \geq 7 \\ 0 & \text{if } t < 7 \end{cases}$$

(a) $\boxed{\text{10 points}}$ Compute F_X, the cumulative distribution of X.

(b) $\boxed{\text{10 points}}$ Assume that we have a repair schedule. Find a time T^* such that if we replace the component at time T^*, the component will still be working with probability 0.99.

1. (a) same 2 chips in a row.
   (b) $\frac{10}{10^2}$.
   (c) no repetitions in first two chips, but the third chip is either the first or second chip.
   (d)
$$\mathbb{P}\{X=3\}=\frac{10\cdot 2\cdot 9}{10^3}$$
   (e) no repetitions in first three chips, but the fourth chip is one of the first three chips.
   (f)
$$\mathbb{P}\{X=4\}=\frac{10\cdot 3\cdot (9)_2}{10^4}$$
   (g) no repetitions in first k-1 chips, but the k-th chip is one of the first k-1 chips.
   (h)
$$\mathbb{P}\{X=k\}=\frac{10\cdot (k-1)\cdot (9)_{k-2}}{10^k}$$
2. (a) $\mathbb{E}[X^2]=27^2+4=731$.
   (b) $\mathbb{E}[Y]=2\mathbb{E}[X]=54$.
3. (a)
$$\mathbb{E}[X]=\frac{1\cdot 1+2\cdot 3+4\cdot 5+10\cdot 3}{12}=\frac{57}{12}.$$
   (b)
$$\mathbb{E}\left[\frac{1}{X}\right]=1\cdot\frac{1}{12}+\frac{1}{2}\cdot\frac{3}{12}+\frac{1}{4}\cdot\frac{5}{12}+\frac{1}{10}\cdot\frac{3}{12}.$$
4. (a)
$$F_X(t)=\begin{cases}0 & \text{if } t<7\\ 1-e^{-5(t-7)} & \text{if } t\ge 7\end{cases}$$
   (b) Want $0.99=\mathbb{P}\{X\ge T^*\}=1-F_X(T^*)$; need $T^*>7$ such that $0.99=e^{-5(T^*-7)}$. In other words,
$$T^*=7-\frac{1}{5}\ln 0.99.$$

Math 361, Section E1, Spring 2003
Exam 3, April 25

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. $\boxed{\text{52 points}}$ Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 4e^{-2t} & \text{if } t \ge s \ge 0 \\ 0 & \text{else} \end{cases}$$

   (a) $\boxed{\text{10 points}}$ Compute f_X, the density of X.

   (b) $\boxed{\text{10 points}}$ Compute f_Y, the density of Y.

   (c) $\boxed{\text{2 points}}$ Are X and Y independent? Yes or no.

   (d) $\boxed{\text{10 points}}$ Compute the conditional density $f_{X|Y}(s|5)$ for all $s \in \mathbb{R}$.

   (e) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X \le 5 \text{ and } Y \le 7\}$.

   (f) $\boxed{\text{10 points}}$ Compute $F_{X,Y}(s,t) \stackrel{\text{def}}{=} \mathbb{P}\{X \le s \text{ and } Y \le t\}$ for all $t \ge s \ge 0$.

2. $\boxed{\text{48 points}}$ Assume that X is a continuous random variable which is exponentially distributed with parameter 2; i.e., it has density

$$f_X(t) = \begin{cases} 2e^{-2t} & \text{if } t \ge 0 \\ 0 & \text{else} \end{cases}$$

   Define now $Y \stackrel{\text{def}}{=} e^{-2X}$.

   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X \ge 5\}$

   (b) $\boxed{\text{10 points}}$ Compute F_X, the cumulative distribution function of X.

   (c) $\boxed{\text{3 points}}$ Compute $\mathbb{P}\{Y \le 3\}$.

   (d) $\boxed{\text{3 points}}$ Compute $\mathbb{P}\{Y \le -2\}$.

   (e) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \le 0.2\}$.

   (f) $\boxed{\text{9 points}}$ Compute the cumulative distribution function F_Y of Y.

   (g) $\boxed{\text{3 points}}$ Is Y a continuous random variable? If so, find its density, if not, state why not.

Answers

1. (a)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^{\infty} 4e^{-2t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} 2e^{-2s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

(b)

$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{t} 4e^{-2t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} 4te^{-2t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

(c) No.

(d)

$$f_{X|Y}(s|5) = \frac{f_{X,Y}(s,5)}{f_Y(5)} = \begin{cases} \frac{4e^{-10}}{20e^{-10}} & \text{if } s \in (0,5) \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{1}{5} & \text{if } s \in (0,5) \\ 0 & \text{else} \end{cases}$$

(e)

$$\mathbb{P}\{X \le 5 \text{ and } Y \le 7\} = \int_{s=-\infty}^{5} \int_{t=-\infty}^{7} f_{X,Y}(s,t)ds\,dt = \int_{s=0}^{5} \int_{t=s}^{7} 4e^{-2t}dt\,ds$$
$$= \int_{s=0}^{5} 2(e^{-2s} - e^{-14})ds = 1 - e^{-10} - 10e^{-14}.$$

(f)

$$F_{X,Y}(s,t) = 1 - e^{-2s} - 2se^{-2t}.$$

2. (a)

$$\mathbb{P}\{X \ge 5\} = \int_{t=5}^{\infty} f_X(t)dt = \int_{t=5}^{\infty} 2e^{-2t}dt = e^{-10}.$$

(b)

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-2t} & \text{if } t \ge 0 \end{cases}$$

(c) $\mathbb{P}\{Y \le 3\} = \mathbb{P}\{e^{-2X} \le 3\} = 1$.

(d) $\mathbb{P}\{Y \le -2\} = \mathbb{P}\{e^{-2X} \le -2\} = 0$.

(e) $\mathbb{P}\{Y \le 0.2\} = \mathbb{P}\{-2X \le \ln 0.2\} = \mathbb{P}\left\{X \ge -\frac{1}{2}\ln 0.2\right\} = e^{\ln 0.2} = 0.2$.

(f)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t < 1 \\ 1 & \text{if } \ge 1 \end{cases}$$

(g) Yes, Y is continuous.

$$f_Y(t) = \begin{cases} 1 & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

Math 361, Section E1, Spring 2003
Final, May 14

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. $\boxed{\text{20 points}}$ A pair of dice is rolled until either a 3 or a 6 appears.
   (a) $\boxed{\text{10 points}}$ What is the probability that the 3 appears before the 6 and the first three is on the 17-th toss?
   (b) $\boxed{\text{10 points}}$ What is the probability that the 3 appears before the 6?

2. $\boxed{\text{20 points}}$ Four cards are taken from a standard deck of cards. What is the probability that they are
   (a) $\boxed{\text{10 points}}$ of different face values.
   (b) $\boxed{\text{10 points}}$ of different suits.

3. $\boxed{\text{30 points}}$ (roughly taken from question 31 on p. 57) Suppose that a 3-person basketball team consists of a guard, a center, and a forward. Suppose that there are 5 such teams (i.e., team 1, team 2, team 3, team 4, and team 5). Suppose that we randomly pick 3 players.
   (a) $\boxed{\text{10 points}}$ What is the probability of picking a pre-existing team (i.e., team 1, team 2, team 3, team 4, or team 5)?
   (b) $\boxed{\text{10 points}}$ What is the probability of picking a "playable" team (i.e., a guard, a center, and a forward)?
   (c) $\boxed{\text{10 points}}$ What is the probability that all three chosen players are guards?

4. $\boxed{\text{50 points}}$ Suppose that X and Y are independent Poisson random variables with parameters λ and ν; i.e., they are discrete random variables with probability mass functions

$$p_X(j) = \begin{cases} e^{-\lambda}\frac{\lambda^j}{j!} & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

$$p_Y(j) = \begin{cases} e^{-\nu}\frac{\nu^j}{j!} & \text{if } j \in \{0, 1 \dots\} \\ 0 & \text{else} \end{cases}$$

   and are independent. Define $Z \stackrel{\text{def}}{=} X + Y$.
   (a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Z = 5\}$ (hint: remember the binomial theorem)
   (b) $\boxed{\text{10 points}}$ Compute the probability mass function of Z.
   (c) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[e^{5X}]$
   (d) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[e^{\theta X}]$ for all $\theta \in \mathbb{R}$.
   (e) $\boxed{\text{10 points}}$ Compute $\mathbb{E}[X]$.

5. $\boxed{\text{30 points}}$ Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 2e^{-2s-t} & \text{if } s \geq 0 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

(a) $\boxed{\text{15 points}}$ Compute $\mathbb{P}\{X < Y\}$

(b) $\boxed{\text{15 points}}$ Compute $\mathbb{P}\{X \leq 5\}$.

Answers

1. Let $p_3 \stackrel{\text{def}}{=} \frac{2}{36}$ be the probability of throwing a 3 and let $p_6 \stackrel{\text{def}}{=} \frac{5}{36}$ be the probability of throwing a 6.
   (a) $(1-p_3-p_6)^{16}p_3$.
   (b) $\sum_{j=1}^{\infty}(1-p_3-p_6)^{j-1}p_3 = \frac{p_3}{p_3+p_6}$.

2. (a)
$$\frac{\binom{13}{4}4^4}{\binom{52}{4}}.$$
   (b)
$$\frac{(13)^4}{\binom{52}{4}}.$$

3. Define $q \stackrel{\text{def}}{=} 1/\binom{15}{3}$.
   (a) $5q$.
   (b) 5^3q.
   (c) $\binom{5}{3}q$.

4. (a)
$$\begin{aligned}\mathbb{P}\{Z=5\} = \sum_{j=0}^{5} p_X(j)p_Y(5-j) &= e^{-\lambda-\nu}\sum_{j=0}^{5}\frac{\lambda^j}{j!}\frac{\nu^{5-j}}{(5-j)!} \\ &= \frac{1}{5!}e^{-\lambda-\nu}\sum_{j=0}^{5}\binom{5}{j}\lambda^j\nu^{5-j} = \frac{(\lambda+\nu)^5}{5!}e^{-\lambda-\nu}.\end{aligned}$$
   (b)
$$p_X(j) = \begin{cases} e^{-(\lambda+\nu)}\frac{(\lambda+\nu)^j}{j!} & \text{if } j \in \{0,1\dots\} \\ 0 & \text{else}\end{cases}$$
   (c)
$$\mathbb{E}[e^{5X}] = \sum_{j=0}^{\infty} e^{-\lambda}e^{5j}\frac{\lambda^j}{j!} = e^{-\lambda}\sum_{j=0}^{\infty}\frac{(\lambda e^5)^j}{j!} = e^{-\lambda}\exp\left[\lambda e^5\right] = \exp\left[\lambda(e^5-1)\right].$$
   (d) $\mathbb{E}[e^{\theta X}] = \exp\left[\lambda\left(e^{\theta}-1\right)\right]$.
   (e) If $\varphi_X(\theta) \stackrel{\text{def}}{=} \mathbb{E}[e^{\theta X}] = \exp\left[\lambda\left(e^{\theta}-1\right)\right]$, then $\varphi'_X(\theta) = \lambda e^{\theta}\exp\left[\lambda\left(e^{\theta}-1\right)\right]$ for all $\theta \in \mathbb{R}$, so $\mathbb{E}[X] = \varphi'_X(0) = \lambda$.

5. (a)

$$\mathbb{P}\{X < Y\} = \int_{s=-\infty}^{\infty} \int_{t=s}^{\infty} f_{X,Y}(s,t) dt\, ds = \int_{s=0}^{\infty} \int_{t=s}^{\infty} 2e^{-2s-t} dt\, ds$$
$$= \int_{s=0}^{\infty} 2e^{-2s} \int_{t=s}^{\infty} e^{-t} dt\, ds = \int_{s=0}^{\infty} 2e^{-2s-s} ds = 2\int_{s=0}^{\infty} e^{-3s} ds = \frac{2}{3}.$$

(b)

$$\mathbb{P}\{X \leq 5\} = \int_{s=-\infty}^{5} \int_{t=-\infty}^{\infty} f_{X,Y}(s,t) dt\, ds = \int_{s=0}^{5} \int_{t=0}^{\infty} 2e^{-2s-t} dt\, ds$$
$$= 2\int_{s=0}^{5} e^{-2s} \int_{t=0}^{\infty} e^{-t} dt\, ds = 2\int_{s=0}^{5} e^{-2s} ds = 1 - e^{-10}.$$