Math 361, Section D1, Spring 2003
Quiz 1, January 29 **Name:** ____________________

1. 10 points Suppose that we have 6 men and 9 women in a room. We want to make a committee of 5. How many ways can we choose the committee so that it consists of 3 men and 2 women?

Answers

1.

$$\binom{6}{3}\binom{9}{2}.$$

Math 361, Section D1, Spring 2003
Quiz 2, February 10 **Name:** ____________________

1. $\boxed{\text{10 points}}$ There are two main cities in a certain state. City A has 1 million people, and city B has 2 million people. Due to smog conditions, 5% of the people in city A will die this year, and 3% of the people in city B will die this year. A person dies. What is the probability that the person was from city A?

Answers

1. Let $A = \{\text{from city A}\}$, $B = \{\text{from city B}\}$, and $D = \{\text{dies this year}\}$. Then we have that

$$\mathbb{P}(A) = \frac{1}{3}, \qquad \mathbb{P}(D|A) = 0.05, \qquad \text{and} \qquad \mathbb{P}(D|B) = 0.03$$

and we can compute that $\mathbb{P}(B) = \frac{2}{3}$. We then have that

$$\mathbb{P}(A|D) = \frac{\mathbb{P}(D \cap A)}{\mathbb{P}(D)} = \frac{\mathbb{P}(D|A)\mathbb{P}(A)}{\mathbb{P}(D|A)\mathbb{P}(A) + \mathbb{P}(D|B)\mathbb{P}(B)} = \frac{(0.05)\frac{1}{3}}{(0.05)\frac{1}{3} + (0.03)\frac{2}{3}}.$$

Math 361, Section D1, Spring 2003
Quiz 3, February 12 **Name:** ____________________

1. 10 points A certain *system* consists of two *machines*, namely machine X and machine Y. Machine Y is a backup; in order for the system to work, at least one of the machines must be working. Assume that both machines are independent, and that the probability that X is working is 0.7, and the probability that Y is working is 0.6. Compute

$$\mathbb{P}\{\text{machine } X \text{ is working} | \text{the system is working}\}.$$

Answers

1. Let $A = \{\text{machine } X \text{ is working}\}$ and $B = \{\text{machine } Y \text{ is working}\}$. Since $A \subset A \cup B$ and by independence,

$$\begin{aligned}\mathbb{P}(A|A\cup B) &= \frac{\mathbb{P}(A)}{\mathbb{P}(A\cup B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B)} \\ &= \frac{\mathbb{P}(A)}{\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A)\mathbb{P}(B)} = \frac{0.7}{0.7+0.6-(0.7)(0.6)} = \frac{0.7}{0.88}.\end{aligned}$$

Math 361, Section D1, Spring 2003
Quiz 4, February 17 **Name:** ______________________

1. 10 points Rebecca and Veronica throw darts. Rebecca hits the target with probability 0.5 and Veronika hits the target with probability 0.7. Suppose that Rebecca and Veronica both independently throw a dart and that the target is hit. What is the probability that both Rebecca and Veronica hit the target?

Answers

1. Let $R = \{\text{Rebecca hits the target}\}$ and $V = \{\text{Veronica hits the target}\}$. Then

$$\mathbb{P}(R\cap V|R\cup V) = \frac{\mathbb{P}(R\cap V)}{\mathbb{P}(R\cup V)} = \frac{\mathbb{P}(R)\mathbb{P}(V)}{\mathbb{P}(R)+\mathbb{P}(V)-\mathbb{P}(R\cap V)} = \frac{\mathbb{P}(R)\mathbb{P}(V)}{\mathbb{P}(R)+\mathbb{P}(V)-\mathbb{P}(R)\mathbb{P}(V)}$$
$$= \frac{(0.5)(0.7)}{0.5+0.7-(0,5)(0.7)}.$$

Math 361, Section D1, Spring 2003
Quiz 5, March 5 **Name:** ____________________

1. 10 points Toss three dice. Let X be the minimum of the faces shown on the dice.
   (a) 2 points Compute $\mathbb{P}\{X \geq 2\}$.
   (b) 3 points Compute $\mathbb{P}\{X \geq i\}$ for all $i \in \{1,2,3,4,5,6\}$.
   (c) 5 points Compute the probability mass function p_X of X.

Answers

1. (a) $\mathbb{P}\{X \geq 2\} = \mathbb{P}\{\text{all dice are} \geq 2\} = \left(\frac{5}{6}\right)^3$.

   (b) $\mathbb{P}\{X \geq i\} = \left(\frac{7-i}{6}\right)^3$.

   (c) $p_X(i) = \mathbb{P}\{X \geq i\} - \mathbb{P}\{X \geq i+1\} = \{(7-i)^3 - (6-i)^3\}/6^3$ for $i \in \{1, 2, 3, 4, 5, 6\}$.

Math 361, Section D1, Spring 2003
Quiz 6, March 17 **Name:** ____________________

1. [10 points] Suppose that a biased coin ($\mathbb{P}\{\mathrm{H}\} = p$) is repeatedly tossed. Let X be the number of tosses until a total of 4 heads has appeared (these 4 heads need not be one after the other).

   (a) [5 points] Verbally explain what the sequence of coin flips looks like if $X = 10$.

   (b) [5 points] Compute $\mathbb{P}\{X = 10\}$.

Answers

1. (a) The 10th flip is a heads, and there are 3 heads in the preceding 9 flips.

   (b) $\binom{9}{3}p^4(1-p)^6$.

Math 361, Section D1, Spring 2003
Quiz 7, April 2 **Name:** ____________________

1. [10 points] Suppose that X is a continuous random variable with uniform density on $(0,1)$; i.e.,

$$f_X(t) = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

Define a new random variable $Y \overset{\text{def}}{=} 2X + 3$.

(a) [2 points] Compute $\mathbb{P}\{Y \le 2\}$.

(b) [2 points] Compute $\mathbb{P}\{Y \le 7\}$.

(c) [3 points] Compute $\mathbb{P}\{Y \le 3.5\}$.

(d) [3 points] Compute the cumulative distribution function F_Y.

Answers

1. (a) $\mathbb{P}\{Y \le 2\} = \mathbb{P}\{2X+3 \le 2\} = \mathbb{P}\{X \le -1/2\} = 0.$

   (b) $\mathbb{P}\{Y \le 7\} = \mathbb{P}\{2X+3 \le 7\} = \mathbb{P}\{X \le 2\} = 1.$

   (c)

$$\begin{aligned}\mathbb{P}\{Y \le 3.5\} = \mathbb{P}\{2X+3 \le 3.5\} &= \mathbb{P}\{X \le 0.25\} \\ &= \int_{t=-\infty}^{0.25} f_X(t)dt = \int_{t=0}^{0.25} dt = 0.25.\end{aligned}$$

   (d)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 3 \\ \frac{t-3}{2} & \text{if } 3 \le t < 5 \\ 1 & \text{if } t \ge 5. \end{cases}$$

Math 361, Section D1, Spring 2003
Quiz 8, April 9 **Name:** ______________________

1. [10 points] Suppose that X is a continuous random variable with moment generating function

$$\mathbb{E}\left[e^{\theta X}\right] = e^{\theta^2+9\theta} \qquad \theta \in \mathbb{R}$$

Define $Y \stackrel{\text{def}}{=} 7X$. Compute $\mathbb{E}\left[e^{\theta Y}\right]$ for all $\theta \in \mathbb{R}$.

Answers

1.

$$\mathbb{E}\left[e^{\theta Y}\right] = \mathbb{E}[\exp[\theta(7X)]] = \mathbb{E}[\exp[(7\theta)X]] = \exp\left[(7\theta)^2 + 9(7\theta)\right]$$
$$= \exp\left[49\theta^2 + 63\theta\right]$$

for all $\theta \in \mathbb{R}$.

Math 361, Section D1, Spring 2003
Quiz 9, April 14 **Name:** ____________________

1. 10 points Suppose that X is a standard Gaussian random variable. i.e., it is continuous with density

$$f_X(t) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{t^2}{2}\right] \qquad t \in \mathbb{R}$$

Define $Y \stackrel{\text{def}}{=} -7X + 5$.

(a) 3 points Compute $\mathbb{P}\{Y \le 40\}$ (in terms of f_X).

(b) 2 points Compute the cumulative distribution function F_Y (in terms of f_X).

(c) 5 points Compute the density f_Y of Y (the answer should be explicit).

Answers

1. (a)

$$\mathbb{P}\{Y \le 40\} = \mathbb{P}\{Y \ge -5\} = \int_{t=-5}^{\infty} f_X(s)ds$$

(b)

$$F_Y(t) = \mathbb{P}\{Y \le t\} = \mathbb{P}\left\{Y \ge \frac{5-t}{7}\right\} = \int_{s=(5-t)/7}^{\infty} f_X(s)ds$$

(c)

$$f_Y(t) = \frac{1}{\sqrt{2\pi(49)}} \exp\left[-\frac{(t-5)^2}{2(49)}\right] \qquad t \in \mathbb{R}$$

Math 361, Section D1, Spring 2003
Quiz 10, April 16 **Name:** ______________________

1. [10 points] Suppose that X is a continuous random variable which is uniformly distributed over the interval $(0,10)$; i.e., it has density

$$f_X(t) = \begin{cases} \frac{1}{10} & \text{if } t \in (0,10) \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \max\{4.5, X\}$.

(a) [3 points] Compute $\mathbb{P}\{Y \le 2.7\}$.

(b) [2 points] Compute $\mathbb{P}\{Y \le 7\}$

(c) [5 points] Compute the cumulative distribution function F_Y of Y.

Answers

1. (a) $\mathbb{P}\{Y \le 2.7\} = \mathbb{P}(\emptyset) = 0$.

   (b) $\mathbb{P}\{Y \le 7\} = \mathbb{P}\{X \le 7\} = \frac{7}{10}$.

   (c)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 4.5 \\ \frac{t}{10} & \text{if } 4.5 \le t < 10 \\ 1 & \text{if } t \ge 10 \end{cases}$$

Math 361, Section D1, Spring 2003
Quiz 11, April 21 **Name:** ______________________

1. Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) \stackrel{\text{def}}{=} \begin{cases} 6t & \text{if } s \ge 0,\ t \ge 0, \text{ and } s+t \le 1 \\ 0 & \text{else} \end{cases}$$

Compute the density f_X of X.

Answers

1.

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=0}^{1-s} 6tdt & \text{if } s \in (0,1) \\ 0 & \text{else} \end{cases} = \begin{cases} 3(1-s)^2 t & \text{if } s \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section D1, Spring 2003
Quiz 12, April 23 Name: ____________________

1. [10 points] Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 2e^{-2(t-s)} & \text{if } t \ge s \text{ and } 0 \le s \le 1 \\ 0 & \text{else} \end{cases}$$

   (a) [5 points] Compute $f_Y(5)$, where f_Y is the density of Y.

   (b) [5 points] Compute the conditional density $f_{X|Y}(s|5)$ for all $s \in \mathbb{R}$.

Answers

1. (a)

$$f_Y(5) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,5)ds = \int_{s=0}^{1} 2e^{-2(5-s)}ds = e^{-10}\left(e^2-1\right).$$

(b)

$$f_{X|Y}(s|5) = \frac{f_{X,Y}(s,5)}{f_Y(5)} = \begin{cases} \frac{2e^{-2(5-s)}}{e^{-10}(e^2-1)} & \text{if } s \in (0,1) \\ 0 & \text{else} \end{cases}$$

Math 361, Section D1, Spring 2003
Exam 1, February 21

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] A certain *system* consists of five *machines*. These machines are hooked up in parallel; in order for the system to work, at least one of the machines must be working. Assume that each machine is independent and works with probability p.
   (a) [10 points] Compute the probability that the system is working.
   (b) [10 points] Compute the probability that exactly two machines are working given that the system is working.

2. [20 points] An insurance company categorizes people as either good risks or bad risks. Of the people who are good risks, 3% are involved in an accident in a given year. Of the people who are bad risks, 10% are involved in an accident in a given year. Assume that in a city of 1 million people, 80% are good risks, and 20% are bad risks.
   (a) [10 points] How many accidents should occur in a given year?
   (b) [10 points] If a randomly selected person did not have an accident last year, what is the probability that they are a good risk driver?

3. [10 points] Urn A has 7 white balls and 4 black balls. Urn B has 3 white balls and 5 black balls. Flip a fair coin; if you get heads, you pick two balls from Urn A, and if you get tails, you pick two balls from Urn B. Suppose that exactly one white ball was selected. What is the probability you flipped a heads?

4. [40 points] Suppose that we flip 5 independent coins. For each coin, $\mathbb{P}\{\text{heads}\} = p$ and $\mathbb{P}\{\text{tails}\} = 1 - p$.
   (a) [10 points] Compute $\mathbb{P}\{\text{HHHHT}\}$.
   (b) [10 points] Compute $\mathbb{P}\{\text{exactly one head}\}$.
   (c) [10 points] Compute $\mathbb{P}\{\text{HTTHH}\}$.
   (d) [10 points] Compute $\mathbb{P}\{\text{exactly two heads}\}$.

5. [10 points] Two brothers are on the same team on a game show. Both brothers independently know the answer to any given question with probability p. They decide upon the following strategy. When they are given a question, they flip a fair coin. If the coin comes up tails, the younger brother answers the question, and if the coin comes up heads, the older brother answers the question. What is the probability that their team answers a specific question right?

Answers

1. (a) $1-(1-p)^5$

   (b)
$$\frac{\binom{5}{2}(1-p)^3p^2}{1-(1-p)^5}.$$

2. (a)
$$0.03(800,000)+0.10(200,000)$$

   (b)
$$\frac{(0.97)(0.80)}{(0.97)(0.80)+(0.90)(0.20)}.$$

3.
$$\frac{\frac{1}{2}(7)(4)/\binom{11}{2}}{\frac{1}{2}(7)(4)/\binom{11}{2}+\frac{1}{2}(3)(5)/\binom{8}{2}}.$$

4. (a) $p^4(1-p)$.

   (b) $5p(1-p)^4$.

   (c) $p^2(1-p)^3$.

   (d) $\binom{5}{2}p^3(1-p)^2$.

5. $\frac{1}{2}p+\frac{1}{2}p=p$.

Math 361, Section D1, Spring 2003
Exam 2, March 21

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [20 points] (Essentially Question 14 on p. 181). A family has n children with probabilty $(1-p)^n p$, for each $n \in \{0, 1 \dots\}$, where p is some parameter. Each child is equally likely to be a boy or a girl.
   (a) [10 points] Compute the probability that the family has 4 boys and 6 girls.
   (b) [10 points] Let X be the number of boys. Compute $\mathbb{P}\{X = 4\}$.

2. [30 points] (Essentially Question 70 on p. 179). A biased coin ($\mathbb{P}\{\text{H}\} = p$) is continually flipped until a heads appears for the 5th time. Let X denote the number of tails that occurs.
   (a) [5 points] Verbally describe what happens when $X = 0$.
   (b) [5 points] Compute $\mathbb{P}\{X = 0\}$.
   (c) [4 points] Verbally describe what happens when $X = 1$.
   (d) [4 points] Compute $\mathbb{P}\{X = 1\}$.
   (e) [3 points] Verbally describe what happens when $X = 2$.
   (f) [3 points] Compute $\mathbb{P}\{X = 2\}$.
   (g) [3 points] Verbally describe what happens when $X = k$, for general $k \in \{0, 1 \dots\}$.
   (h) [3 points] Compute $\mathbb{P}\{X = k\}$.

3. [20 points] The lifetime X of a computer, measured in years, is a continuous random variable with density

$$f_X(t) = \begin{cases} \frac{6}{t^2} & \text{if } t \geq 6 \\ 0 & \text{if } t < 6 \end{cases}$$

   (a) [10 points] Compute $\mathbb{P}\{X > 35\}$.
   (b) [10 points] Compute the cumulative distribution function of X.

4. [30 points] Suppose that the random variable X has cumulative distribution function

$$F_X(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{4} & \text{if } 0 \leq t < 1 \\ \frac{t}{3} & \text{if } 1 \leq t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}$$

   Compute

(a) 10 points $\mathbb{P}\{X \leq 1\}$

(b) 10 points $\mathbb{P}\{X < 1\}$.

(c) 10 points $\mathbb{P}\{0 < X < 2\}$.

Answers

1. (a)
$$\binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6(1-p)^{10}p = \binom{10}{4}\left(\frac{1}{2}\right)^{10}(1-p)^{10}p.$$
   (b)
$$\sum_{j=4}^{\infty}\binom{j}{4}\left(\frac{1}{2}\right)^j(1-p)^j p.$$

2. (a) 5 heads.
   (b) p^5.
   (c) a head on 6th toss, and 4 heads in the first 5 flips.
   (d) $\binom{5}{4}p^4(1-p)p = \binom{5}{4}p^5(1-p)$.
   (e) a tail on 7th toss, and 4 heads in the first 6 flips.
   (f) $\binom{6}{4}p^4(1-p)^2p = \binom{6}{4}p^5(1-p)^2$.
   (g) a tail on $5+k$-th toss, and 4 heads in the first $4+k$ flips.
   (h) $\binom{4+k}{4}p^4(1-p)^kp = \binom{4+k}{4}p^5(1-p)^k$.

3. (a)
$$\mathbb{P}\{X > 35\} = \int_{35}^{\infty}\frac{6}{t^2}dt = \frac{6}{35}.$$
   (b)
$$F_X(t) = \begin{cases} 0 & \text{if } t < 6 \\ 1-\frac{6}{t} & \text{if } t \ge 6 \end{cases}$$

4. (a) $\mathbb{P}\{X \le 1\} = F_X(1) = \frac{1}{3}$
   (b) $\mathbb{P}\{X < 1\} = F_X(1-) = \frac{1}{4}$
   (c) $\mathbb{P}\{0 < X < 2\} = F_X(2-) - F_X(0) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$.

Math 361, Spring 2003
Exam 2 (Makeup), March 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [40 points] (Essentially Question 30 on p. 183). A jar contains 10 distinct chips. You successively draw chips from the jar, replacing the chip at each turn. Let X denote the number of draws until you select a chip which you had previously selected.
   (a) [6 points] Verbally describe what happens when $X = 2$.
   (b) [6 points] Compute $\mathbb{P}\{X = 2\}$.
   (c) [5 points] Verbally describe what happens when $X = 3$.
   (d) [5 points] Compute $\mathbb{P}\{X = 3\}$.
   (e) [4 points] Verbally describe what happens when $X = 4$.
   (f) [4 points] Compute $\mathbb{P}\{X = 4\}$.
   (g) [3 points] Verbally describe what happens when $X = k$, for general $k \in \{2, 3 \dots\}$.
   (h) [7 points] Compute $\mathbb{P}\{X = k\}$.

2. [20 points] Suppose that X is a random variable with mean 27 and variance 2. Define also $Y \stackrel{\text{def}}{=} 2X$.
   (a) [10 points] Compute $\mathbb{E}[X^2]$.
   (b) [10 points] Compute $\mathbb{E}[Y]$.

3. [20 points] Suppose that X is a discrete random variable with probability mass function

$$p_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{12} & \text{if } j = 1 \\ \frac{3}{12} & \text{if } j = 2 \\ \frac{5}{12} & \text{if } j = 4 \\ \frac{3}{12} & \text{if } j = 10 \end{cases}$$

   (a) [10 points] Compute $\mathbb{E}[X]$.
   (b) [10 points] Compute $\mathbb{E}\left[\frac{1}{X}\right]$.

4. [20 points] Suppose that the lifetime of a certain computer component is a continuous random variable with density

$$f_X(t) = \begin{cases} 5e^{-5(t-7)} & \text{if } t \geq 7 \\ 0 & \text{if } t < 7 \end{cases}$$

(a) 10 points Compute F_X, the cumulative distribution of X.

(b) 10 points Assume that we have a repair schedule. Find a time T^* such that if we replace the component at time T^*, the component will still be working with probability 0.99.

Answers

1. (a) same 2 chips in a row.
   (b) $\frac{10}{10^2}$.
   (c) no repetitions in first two chips, but the third chip is either the first or second chip.
   (d)
$$\mathbb{P}\{X = 3\} = \frac{10 \cdot 2 \cdot 9}{10^3}$$
   (e) no repetitions in first three chips, but the fourth chip is one of the first three chips.
   (f)
$$\mathbb{P}\{X = 4\} = \frac{10 \cdot 3 \cdot (9)_2}{10^4}$$
   (g) no repetitions in first k-1 chips, but the k-th chip is one of the first k-1 chips.
   (h)
$$\mathbb{P}\{X = k\} = \frac{10 \cdot (k-1) \cdot (9)_{k-2}}{10^k}$$
2. (a) $\mathbb{E}[X^2] = 27^2 + 4 = 731$.
   (b) $\mathbb{E}[Y] = 2\mathbb{E}[X] = 54$.
3. (a)
$$\mathbb{E}[X] = \frac{1 \cdot 1 + 2 \cdot 3 + 4 \cdot 5 + 10 \cdot 3}{12} = \frac{57}{12}.$$
   (b)
$$\mathbb{E}\left[\frac{1}{X}\right] = 1 \cdot \frac{1}{12} + \frac{1}{2} \cdot \frac{3}{12} + \frac{1}{4} \cdot \frac{5}{12} + \frac{1}{10} \cdot \frac{3}{12}.$$
4. (a)
$$F_X(t) = \begin{cases} 0 & \text{if } t < 7 \\ 1 - e^{-5(t-7)} & \text{if } t \geq 7 \end{cases}$$
   (b) Want $0.99 = \mathbb{P}\{X \geq T^*\} = 1 - F_X(T^*)$; need $T^* > 7$ such that $0.99 = e^{-5(T^*-7)}$. In other words,
$$T^* = 7 - \frac{1}{5} \ln 0.99.$$

Math 361, Section D1, Spring 2003
Exam 3, April 25

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 100 Points

1. [50 points] Consider a transformation

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 10 & \text{if } 0 < u < 1 \\ 5 & \text{if } 1 \le u < 2 \\ u-2 & \text{if } 2 \le u < 3 \end{cases}$$

Let X be a continuous random variable which is uniformly distributed on $(0,3)$; i.e.,

$$f_X(t) = \begin{cases} \frac{1}{3} & t \in (0,3) \\ 0 & \text{else} \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

(a) [10 points] Graph the function φ

(b) [10 points] Compute $\mathbb{P}\{Y \le 7\}$.

(c) [10 points] Compute $\mathbb{P}\{Y \le 4\}$.

(d) [10 points] Compute $\mathbb{P}\{Y \le 0.5\}$.

(e) [10 points] Compute the cumulative distribution function F_Y of Y.

2. [28 points] Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 8(t-s)e^{-2t} & \text{if } t \ge s \ge 0 \\ 0 & \text{else} \end{cases}$$

(a) [10 points] Compute f_X, the density of X (recall that $\int_0^\infty te^{-t}dt = 1$).

(b) [10 points] Compute f_Y, the density of Y.

(c) [3 points] Are X and Y independent? Yes or no.

(d) [5 points] Compute the conditional density $f_{X|Y}(s|5)$ for all $s \in \mathbb{R}$.

3. [22 points] Suppose that X has moment generating function

$$\varphi_X(\theta) = \mathbb{E}[e^{\theta X}] = \exp\left[\frac{25}{2}\theta^2 + 7\theta\right]. \qquad \theta \in \mathbb{R}$$

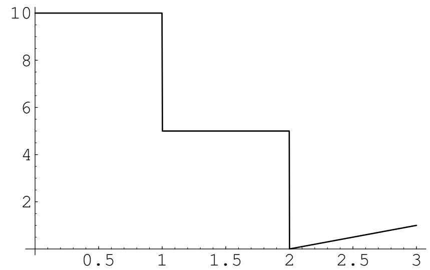
(a) [10 points] What is the density of X?

(b) [7 points] What is the mean of X?

(c) [5 points] What is the variance of X?

Answers

1. (a) See figure.



(b) $\mathbb{P}\{Y \le 7\} = \mathbb{P}\{1 \le X < 3\} = \frac{2}{3}$.

(c) $\mathbb{P}\{Y \le 4\} = \mathbb{P}\{2 \le X < 3\} = \frac{1}{3}$.

(d) $\mathbb{P}\{Y \le 0.5\} = \mathbb{P}\{2 \le X \le 2.5\} = \frac{1}{6}$.

(e)
$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t}{3} & \text{if } 0 \le t < 1 \\ \frac{1}{3} & \text{if } 1 \le t < 5 \\ \frac{2}{3} & \text{if } 5 \le t < 10 \\ 1 & \text{if } t \ge 10 \end{cases}$$

2. (a)
$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s,t)dt = \begin{cases} \int_{t=s}^{\infty} 8(t-s)e^{-2t}dt & \text{if } s > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} 2e^{-2s} & \text{if } s > 0 \\ 0 & \text{else} \end{cases}$$

(b)
$$f_Y(t) = \int_{s=-\infty}^{\infty} f_{X,Y}(s,t)ds = \begin{cases} \int_{s=0}^{t} 8(t-s)e^{-2t}ds & \text{if } t > 0 \\ 0 & \text{else} \end{cases} = \begin{cases} 4t^2e^{-2t} & \text{if } t > 0 \\ 0 & \text{else} \end{cases}$$

(c) No.

(d)
$$f_{X|Y}(s|5) = \frac{f_{X,Y}(s,5)}{f_Y(5)} \begin{cases} \frac{8(5-s)e^{-10}}{100e^{-10}} & \text{if } s \in (0,5) \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{2}{25}(5-s) & \text{if } s \in (0,5) \\ 0 & \text{else} \end{cases}$$

3. (a)
$$f_X(t) = \frac{1}{\sqrt{50\pi}} \exp\left[-\frac{(t-7)^2}{50}\right] \qquad t \in \mathbb{R}$$

(b) $\mathbb{E}[X] = 7$.

(c) 25

Math 361, Section D1, Spring 2003
Final, May 16

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.
Maximum possible score: 150 Points

1. [40 points] Suppose that we have a *red* coin and a *blue* coin. Suppose that both coins are biased; $\mathbb{P}\{\text{red heads}\} = p$ and $\mathbb{P}\{\text{blue tails}\} = q$, where p and q are some parameters. Suppose now that we repeatedly flip both coins (at the same time, one in the left hand and one in the right hand) until the coins disagree. Let N denote the first time that both coins disagree (i.e., if $N = 2$, then both coins agree on the first flip and then disagree on the second flip). Let $X = H$ if the red coin shows heads when the coins first disagree (i.e., on the N-th flip), and let $X = T$ if the red coin shows tails when the coins first disagree (i.e., on the N-th flip).
   (a) [10 points] For a single flip, compute the probability that the coins disagree
   (b) [10 points] Compute $\mathbb{P}\{N = 3\}$.
   (c) [10 points] Compute $\mathbb{P}\{X = H \text{ and } N = 3\}$.
   (d) [10 points] Compute $\mathbb{P}\{X = H\}$. Compute the answer explicitly.

2. [40 points] (roughly taken from problem 11 on page 185) Teams A and B play a series of independent games against each other. The first team to win 4 games is declared the winner of the series. Suppose that in each game, the probability that A wins is p (and consequently the probability that B wins is $1 - p$).
   (a) [10 points] What is the probability that A wins the series in exactly 7 games?
   (b) [10 points] What is the probability that A wins the series ?
   (c) [10 points] Compute the probability that A wins the first game and wins the series.
   (d) [10 points] Compute the probability that A wins the series given that it wins the first game.

3. [10 points] Suppose that a continuous random variable X has density

$$f_X(t) = \begin{cases} 2(1-t) & \text{if } t \in (0,1) \\ 0 & \text{else} \end{cases}$$

   Compute $\mathbb{E}[X]$.

4. [60 points] Suppose that X and Y are independent exponential random variables with parameters 2 and 7; i.e., they are both continuous random variables with densities

$$f_X(t) = \begin{cases} 2e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$f_Y(t) = \begin{cases} 7e^{-7t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

and are independent. Define

$$Z \stackrel{\text{def}}{=} \min\{X, 3Y\}.$$

(a) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{X \geq 13\}$.

(b) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Y \geq t\}$ for all $t \geq 0$.

(c) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Z \geq 4\}$ (hint: you first should clearly understand what it means in terms of X and Y when $Z \geq 4\}$.

(d) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Z \geq t\}$ for all $t \geq 0$.

(e) $\boxed{\text{10 points}}$ Compute the cumulative distribution of Z.

(f) $\boxed{\text{10 points}}$ Compute $\mathbb{P}\{Z \geq 4 \text{ and } X \leq 20\}$.

Answers

1. (a) $\mathbb{P}\{\text{red heads and blue tails}\} + \mathbb{P}\{\text{blue heads and red tails}\} = pq + (1-p)(1-q)$.
   (b) $\mathbb{P}\{N = 3\} = (1 - pq - (1-p)(1-q))^2(pq + (1-p)(1-q))$.
   (c) $\mathbb{P}\{X = H \text{ and } N = 3\} = (1 - pq - (1-p)(1-q))^2 pq$.
   (d)

$$\mathbb{P}\{X = H\} = \sum_{j=1}^{\infty} \mathbb{P}\{X = H \text{ and } N = j\}$$
$$= \sum_{j=1} (1 - pq - (1-p)(1-q))^{j-1} pq = \frac{pq}{pq + (1-p)(1-q)}.$$

2. (a) The 7th game must be won by A. In the preceding 6 games, A must win 3 of them (and B must win 3). The answer is $\binom{6}{3}p^4(1-p)^3$.
   (b) If A wins, it must do so in at least 4 games and in at most 7 games. The answer is

$$\sum_{j=4}^{7} \binom{j-1}{2} p^4 (1-p)^{j-4}.$$

   game and wins the series.
   (c) $\sum_{j=4}^{7} \binom{j-2}{2} p^4 (1-p)^{j-4}$.
   (d) $\sum_{j=4}^{7} \binom{j-2}{2} p^4 (1-p)^{j-5}$.

3.

$$\mathbb{E}[X] = \int_{t=-\infty}^{\infty} t f_X(t)dt = 2\int_0^1 t(1-t)dt$$
$$= 2\int_0^1 \{t - t^2\}dt = 2\left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Bigg|_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

4. (a)

$$\mathbb{P}\{X \ge 13\} = \int_{t=13}^{\infty} f_X(t)dt = \int_{t=13}^{\infty} 2e^{-2t}dt = e^{-26}.$$

   (b) For $t \ge 0$,

$$\mathbb{P}\{Y \ge t\} = e^{-7t}.$$

   (c)

$$\mathbb{P}\{Z \ge 4\} = \mathbb{P}\{X \ge 4 \text{ and } Y \ge 4/3\} = \mathbb{P}\{X \ge 4\}\mathbb{P}\{Y \ge 4/3\} = e^{-8}e^{-28/3}.$$

   (d) For $t \ge 0$,

$$\mathbb{P}\{Z \ge t\} = \mathbb{P}\{X \ge t\}\mathbb{P}\{Y \ge t/3\} = e^{-2t-7t/3}.$$

(e)

$$F_Z(t) = \begin{cases} 1 - \exp\left[-(2+7/3)t\right] & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

(f)

$$\begin{aligned} \mathbb{P}\{Z \geq 4 \text{ and } X < 20\} &= \mathbb{P}\{4 \leq X < 20\}\mathbb{P}\{Y \geq 4/3\} \\ &= (\mathbb{P}\{X \geq 4\} - \mathbb{P}\{X \geq 20\})\,\mathbb{P}\{Y \geq 4/3\} = \left(e^{-8} - e^{40}\right) e^{-28/3}. \end{aligned}$$