

Math 361, Section F1, Spring 2002
Quiz 1, January 18

Name: _____

1. 10 points Assume that

$$\mathbb{P}(A) = 0.5, \quad \mathbb{P}(B) = 0.6 \quad \text{and} \quad \mathbb{P}(B \setminus A) = 0.1.$$

(a) 5 points Compute $\mathbb{P}(A \cup B)$

(b) 5 points Compute $\mathbb{P}(A \setminus B)$.

ANSWERS

1. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A) = 0.5 + 0.1 = 0.6.$

2. $\mathbb{P}(A \setminus B) = \mathbb{P}(A \cup B) - \mathbb{P}(B) = 0.6 - 0.6 = 0.$

Math 361, Section F1, Spring 2002
Quiz 2, January 25

Name: _____

1. 10 points Assume that

$$\mathbb{P}(A) = 0.5, \quad \mathbb{P}(B) = 0.6 \quad \text{and} \quad \mathbb{P}(B \setminus A) = 0.1.$$

(a) 7 points Compute $\mathbb{P}(A|B)$

(b) 3 points Are A and B independent?

ANSWERS

1.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A) + \mathbb{P}(B \setminus A)}{\mathbb{P}(B)} = \frac{0.5 + 0.1}{0.6} = 1.$$

2. No, since $\mathbb{P}(A|B) = 1 \neq 0.5 = \mathbb{P}(A)$.

1. 10 points Suppose that cards are dealt from an ordinary deck of playing cards one at a time until the first king appears. Find the probability that this occurs on the 7th card dealt. (Remember that the deck has 52 cards, 4 of which are kings.)

ANSWERS

1.

$$\frac{(48)_6 \cdot 4}{(52)_7}.$$

1. 10 points Suppose a random variable X has density f_X , where

$$f_X(0) = 0.1, \quad f_X(1) = 0.5, \quad f_X(2) = 0.05, \quad f_X(3) = .3, \quad \text{and} \quad f_X(4) = 0.05$$

- (a) 5 points Compute $\mathbb{P}\{X \text{ is odd}\}$.
- (b) 5 points Compute $\mathbb{P}\{X \leq 2.5\}$.

ANSWERS

1. $\mathbb{P}\{X \text{ is odd}\} = f_X(1) + f_X(3) = 0.5 + 0.3 = 0.8.$
2. Compute $\mathbb{P}\{X \leq 2.5\} = f_X(0) + f_X(1) + f_X(2) = 0.1 + 0.5 + 0.05 = 0.65.$

Math 361, Section F1, Spring 2002
Quiz 5, February 13

Name: _____

1. 10 points Explicitly evaluate

$$\sum_{n=6}^{\infty} \left(\frac{1}{3}\right)^{n-4} \left(\frac{4}{10}\right).$$

ANSWERS

1.

$$\sum_{n=6}^{\infty} \left(\frac{1}{3}\right)^{n-4} \left(\frac{4}{10}\right) = \left(\frac{4}{10}\right) \left(\frac{1}{3}\right)^2 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{4}{10} \cdot \frac{1}{3}}{1 - \frac{1}{3}}.$$

1. 10 points Let X_1, X_2, X_3 and X_4 be independent random variables, all of which are geometrically distributed with parameter p . Define

$$Y \stackrel{\text{def}}{=} \min\{X_1, X_2, X_3, X_4\}.$$

Find the density of Y . Note: the fact that the X_i 's are independent means that

$$\mathbb{P}\{X_1 = j_1, X_2 = j_2, X_3 = j_3, X_4 = j_4\} = \mathbb{P}\{X_1 = j_1\}\mathbb{P}\{X_2 = j_2\}\mathbb{P}\{X_3 = j_3\}\mathbb{P}\{X_4 = j_4\}$$

for all j_1, j_2, j_3 and j_4 .

ANSWERS

1. $\mathbb{P}\{Y = k\} = 0$ if $k \leq 0$. For $k \geq 0$ an integer,

$$\mathbb{P}\{X_i \geq k\} = \sum_{j=k}^{\infty} p(1-p)^{j-1} = (1-p)^{j-1};$$

thus,

$$\mathbb{P}\{Y \geq k\} = \prod_{i=1}^4 \mathbb{P}\{X_k \geq k\} = \{(1-p)^4\}^{j-1}$$

hence

$$\begin{aligned} \mathbb{P}\{Y = k\} &= \mathbb{P}\{Y \geq k\} - \mathbb{P}\{Y \geq k+1\} = \{(1-p)^4\}^{k-1} - \{(1-p)^4\}^k \\ &= \{(1-p)^4\}^{k-1} \{1 - \{(1-p)^4\}\}. \end{aligned}$$

Thus, Y is geometric with parameter $1 - (1-p)^4$.

1. 10 points Let X and Y be independent geometric random variables with parameter p . Define $Z \stackrel{\text{def}}{=} X + Y$. Find the density of Z .

ANSWERS

1.

$$\begin{aligned} f_Z(j) &= \sum_k f_X(k) f_Y(j-k) = \sum_{k=-\infty}^{\infty} \chi_{\{k \geq 1\}} \chi_{\{j-k \geq 1\}} p^2 (1-p)^{j-1} (1-p)^{j-k-1} \\ &= p^2 (1-p)^{k-2} \sum_{k=-\infty}^{\infty} \chi_{\{1 \leq k \leq j-1\}} = \begin{cases} (j-1)p^2(1-p)^{k-2} & \text{if } j \geq 2 \\ 0 & \text{else} \end{cases} \end{aligned}$$

1. 10 points Let X a random variable with density

$$f_X(0) = \frac{1}{8}, \quad f_X(1) = \frac{2}{8}, \quad f_X(2) = \frac{1}{8}, \quad f_X(3) = \frac{1}{8}, \quad f_X(4) = \frac{3}{8}$$

- (a) 4 points Compute $\mathbb{E}[X]$.
- (b) 4 points Compute $\mathbb{E}[X^2]$.
- (c) 2 points Compute the variance σ^2 of X .

ANSWERS

1. (a) $\mathbb{E}[X] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{2}{8} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{3}{8} = \frac{19}{8}$.
- (b) $\mathbb{E}[X] = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{2}{8} + 2^2 \cdot \frac{1}{8} + 3^2 \cdot \frac{1}{8} + 4^2 \cdot \frac{3}{8} = \frac{63}{8}$.
- (c) $\frac{63}{8} - \left(\frac{19}{8}\right)^2$.

1. 10 points Let X and Y be independent discrete random variables with densities

$$f_X(j) = \begin{cases} p(1-p)^j & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$
$$f_Y(j) = \begin{cases} (j+1)p^2(1-p)^j & \text{if } j \in \{0, 1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

Let $Z \stackrel{\text{def}}{=} X + Y$. Compute $f_Z(3)$.

ANSWERS

1.

$$\begin{aligned} f_Z(3) &= \sum_k f_X(3-k)f_Y(k) = \sum_{k=-\infty}^{\infty} \chi_{\{3-k \geq 0\}} \chi_{\{k \geq 0\}} p(1-p)^{3-j}(j+1)p^2(1-p)^j \\ &= p^3(1-p)^3 \sum_{k=0}^3 (j+1) = p^3(1-p)^3 \sum_{k=1}^4 j \\ &= p^3(1-p)^3 \frac{4 \cdot 5}{2} = 10p^3(1-p)^3 \end{aligned}$$

1. 10 points Let X be a continuous exponential random variable with parameter 5; i.e.,

$$f_X(t) = \begin{cases} 5e^{-5t} & \text{if } t \geq 0 \\ 0 & \text{else} \end{cases}$$

- (a) 5 points Compute $\mathbb{P}\{X \geq 10\}$.
- (b) 5 points Compute $\mathbb{P}\{2 \leq X \leq 70 | X \geq 10\}$.

ANSWERS

1. $\mathbb{P}\{X \geq 10\} = \int_{t=10}^{\infty} 5e^{-5t} dt = e^{-50}$.

2.

$$\begin{aligned} \mathbb{P}\{2 \leq X \leq 70 | X \geq 10\} &= \frac{\mathbb{P}\{10 \leq X \leq 70\}}{\mathbb{P}\{X \geq 10\}} \\ &= \frac{\int_{t=10}^{70} 5e^{-5t} dt}{e^{-50}} = \frac{e^{-50} - e^{-350}}{e^{-50}} = 1 - e^{-300}. \end{aligned}$$

Math 361, Section F1, Spring 2002
Quiz 11, April 17

Name: _____

Let X and Y be continuous random variables with joint density

$$f_{X,Y}(s,t) = \begin{cases} 5e^{-5t} & \text{if } 0 \leq s \leq 1 \text{ and } t \geq 0 \\ 0 & \text{else} \end{cases}$$

1. 10 points Compute $\mathbb{P}\{Y \leq X\}$.

ANSWERS

1.

$$\begin{aligned}\mathbb{P}\{Y \leq X\} &= \int_{t=0}^{\infty} \int_{s=0}^{\infty} \chi_{\{t \leq s\}} f_{X,Y}(s, t) ds dt = \int_{s=0}^1 \int_{t=0}^s 5e^{-5t} dt ds \\ &= \int_{s=0}^1 \{1 - e^{-5s}\} ds = 1 - \frac{1}{5}\{1 - e^{-5}\}.\end{aligned}$$

1. 10 points Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} 4st & \text{if } s \geq 1 \text{ and } 0 \leq t \leq \frac{1}{s^2} \\ 0 & \text{else.} \end{cases}$$

- (a) 5 points Compute $f_X(2)$.
- (b) 5 points Compute the conditional density $f_{Y|X}(t|2)$ for all t .

ANSWERS

1. (a)

$$f_X(2) = \int_{t=-\infty}^{\infty} f_{X,Y}(2, t) dt = \int_{t=0}^{1/4} 8t dt = 4t^2 \Big|_{t=0}^{1/4} = \frac{1}{4}.$$

(b)

$$f_{Y|X}(t|2) = \frac{f_{X,Y}(2, t)}{f_X(2)} = \begin{cases} 32t & \text{if } 0 \leq t \leq \frac{1}{4} \\ 0 & \text{else} \end{cases}$$

Math 361, Section F1, Spring 2002
Exam 1, February 20

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

- 20 points** Toss a sequence of independent and identically distributed coins such that $\mathbb{P}\{H\} = 1/8$ for each coin. Let X be the position of the first heads.

 - 5 points** Compute the density of X .
 - 10 points** Compute $\mathbb{P}\{X \geq 7 | X \geq 3\}$.
 - 5 points** Are the sets $\{X \geq 7\}$ and $\{X \geq 3\}$ independent?
- 10 points** Suppose that X is geometric with parameter p and Y is geometric with parameter q . Assume also that X and Y are independent. Compute $\mathbb{P}\{X = Y\}$.
- 15 points** Suppose that X and Y have joint density

		Y			
		0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words, $f_{X,Y}(0, 2) = \frac{3}{27}$.

- 5 points** Compute $\mathbb{P}\{X = 3\}$
 - 5 points** Compute $\mathbb{P}\{X \geq 0\}$
 - 5 points** Compute $\mathbb{P}\{X = 0 | Y \text{ is even}\}$
- 30 points** Suppose that a group of people contains 55 women and 45 men. Suppose that 10% of the men smoke and 15% of the women smoke. Randomly choose a person. If the person is a smoker, what is the probability that you have chosen a male? Please be clear about your calculations. If I don't see any calculations, I will not give you any points. Do not attempt to do any serious multiplication or division.
 - 25 points** A box consists of
 - 5 Alanis Morissette CD's
 - 3 Suzanne Vega CD's
 - 7 Led Zeppelin CD's

You grab 7 CD's from the box.

- 5 points** What is the probability that you grabbed 1 Alanis Morissette CD, 2 Suzanne Vega CD's, and 4 Led Zeppelin CD's?

- (b) 10 points What is the probability that you grabbed at least one Alanis Morissette CD?
- (c) 10 points What is the probability that you grabbed at least one Alanis Morissette CD given that you grabbed exactly 2 Suzanne Vega CD's?

ANSWERS

1. (a)

$$\mathbb{P}\{X = k\} = \begin{cases} \frac{1}{8} \left(\frac{7}{8}\right)^{j-1} & \text{if } j \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$

For $k \in \{1, 2, \dots\}$,

$$\mathbb{P}\{X \geq k\} = \sum_{j=k}^{\infty} \frac{1}{8} \left(\frac{7}{8}\right)^{j-1} = \left(\frac{7}{8}\right)^{k-1}.$$

(b)

$$\mathbb{P}\{X \geq 7 | X \geq 3\} = \frac{\mathbb{P}\{X \geq 7 \text{ and } X \geq 3\}}{\mathbb{P}\{X \geq 3\}} = \frac{\mathbb{P}\{X \geq 7\}}{\mathbb{P}\{X \geq 3\}} = \frac{\left(\frac{7}{8}\right)^6}{\left(\frac{7}{8}\right)^2} = \left(\frac{7}{8}\right)^4.$$

(c) No, since $\mathbb{P}\{X \geq 7 | X \geq 3\} = \left(\frac{7}{8}\right)^4 \neq \left(\frac{7}{8}\right)^6 = \mathbb{P}\{X \geq 7\}$.

2.

$$\begin{aligned} \mathbb{P}\{X = Y\} &= \sum_{k=1}^{\infty} \mathbb{P}\{X = Y = k\} = \sum_{k=1}^{\infty} pq(1-p)^{k-1}(1-q)^{k-1} = pq \sum_{k=0}^{\infty} \{(1-p)(1-q)\}^k \\ &= \frac{pq}{1 - \{(1-p)(1-q)\}} = \frac{pq}{p+q-pq} \end{aligned}$$

3. (a) $\mathbb{P}\{X = 3\} = \frac{3+4+7}{27}$.

(b) $\mathbb{P}\{X \geq 0\} = \frac{6+3+3+3+4}{27}$.

(c) $\mathbb{P}\{X = 0 | Y \text{ is even}\} = \left(\frac{6+3}{27}\right) / \left(\frac{3+6+3+1+3}{27}\right) = \frac{9}{16}$

4.

$$\mathbb{P}\{M|S\} = \frac{\mathbb{P}\{S|M\}\mathbb{P}\{M\}}{\mathbb{P}\{S|M\}\mathbb{P}\{M\} + \mathbb{P}\{S|W\}\mathbb{P}\{W\}} = \frac{(0.1)(0.45)}{(0.1)(0.45) + (0.5)(0.55)}.$$

5. (a)

$$\binom{5}{1} \binom{3}{2} \binom{7}{4} / \binom{15}{7}$$

(b)

$$1 - \binom{10}{7} / \binom{15}{7}$$

(c)

$$\begin{aligned}\mathbb{P}\{\text{at least one Morisette} \mid 2 \text{ Vega}\} &= 1 - \mathbb{P}\{\text{no Morisette} \mid 2 \text{ Vega}\} \\ &= 1 - \frac{\mathbb{P}\{\text{no Morisette and 2 Vega}\}}{\mathbb{P}\{2 \text{ Vega}\}} \\ &= 1 - \left\{ \frac{\binom{3}{2} \binom{7}{5} / \binom{15}{7}}{\binom{3}{2} \binom{12}{5} / \binom{15}{7}} \right\} \\ &= 1 - \frac{\binom{3}{2} \binom{7}{5}}{\binom{3}{2} \binom{12}{5}}.\end{aligned}$$

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 50 points Let X and Y be independent random variables with densities

$$f_X(j) = \begin{cases} (1-p)^{j-1}p & \text{if } j \in \{1, 2, \dots\} \\ 0 & \text{else} \end{cases}$$
$$f_Y(j) = \begin{cases} \frac{1}{4} & \text{if } j = 0 \\ \frac{3}{4} & \text{if } j = 2 \\ 0 & \text{else} \end{cases}$$

Define $Z \stackrel{\text{def}}{=} X + Y$.

- (a) 20 points Compute the density f_Z of Z . Note. If you are so inclined, you might want to write the density of Y as $f_Y(j) = U(j)\chi_{\{0,2\}}(j)$, where $U(0) = \frac{1}{4}$ and $U(2) = \frac{3}{4}$.
- (b) 10 points Compute the moment generating function $\mathbb{E}[e^{\theta X}]$ of X .
- (c) 10 points Compute $\mathbb{E}[X]$.
- (d) 5 points Compute $\mathbb{E}[Y]$.
- (e) 5 points Compute $\mathbb{E}[10X - Y]$.
2. 20 points Suppose that X and Y have joint density

		Y			
		0	1	2	3
	-1	3/27	1/27	1/27	3/27
X	0	6/27	0	3/27	3/27
	3	3/27	0	0	4/27

In other words, $f_{X,Y}(0, 2) = \frac{3}{27}$.

- (a) 7 points Compute $\mathbb{E}[X]$
- (b) 6 points Compute $\mathbb{E}[X^2]$
- (c) 7 points Compute $\mathbb{E}[X^2Y]$

3. 30 points Let X be a geometric random variable with parameter p . Let φ be the function

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} 10 - u & \text{if } u \leq 10 \\ 0 & \text{if } 10 < u \leq 20 \\ u - 20 & \text{if } u > 20 \end{cases}$$

Find the density of $Y \stackrel{\text{def}}{=} \varphi(X)$. Hint: You may want to first graph φ .

ANSWERS

1. (a)

$$\begin{aligned}
 f_Z(j) &= \sum_{k=-\infty}^{\infty} f_X(j-k)f_Y(k) = \frac{1}{4}f_X(j) + \frac{3}{4}f_X(j-2) \\
 &= \begin{cases} 0 & \text{if } j \leq 0 \\ \frac{1}{4}p(1-p)^{j-1} & \text{if } j \geq 1 \text{ and } j-2 \leq 0 \\ \frac{1}{4}p(1-p)^{j-1} + \frac{3}{4}p(1-p)^{j-3} & \text{if } j-2 \geq 1 \end{cases} \\
 &= \begin{cases} 0 & \text{if } j \leq 0 \\ \frac{1}{4}p(1-p)^{j-1} & \text{if } 1 \leq j \leq 2 \\ \frac{1}{4}p(1-p)^{j-1} + \frac{3}{4}p(1-p)^{j-3} & \text{if } j \geq 3 \end{cases}
 \end{aligned}$$

(b)

$$\mathbb{E}[e^{\theta X}] = \sum_{j=1}^{\infty} e^{\theta j} p(1-p)^{j-1} = \frac{pe^{\theta}}{1 - e^{\theta}(1-p)}.$$

(c) By differentiating moment generating function at $\theta = 0$, we get that

$$\mathbb{E}[X] = \frac{e^{\theta}p\{1 - e^{\theta}(1-p)\} + e^{\theta}(1-p)e^{\theta}p}{\{1 - e^{\theta}(1-p)\}^2} = \frac{1}{p}.$$

(d) $\mathbb{E}[Y] = 0\frac{1}{4} + 2\frac{3}{4} = \frac{6}{4} = \frac{3}{2}.$

(e) $\mathbb{E}[10X - Y] = 10\mathbb{E}[X] - \mathbb{E}[Y] = \frac{10}{p} - \frac{3}{2}.$

2. (a) $\mathbb{E}[X] = (-1)\frac{3+1+1+3}{27} + 3\frac{3+4}{27} = \frac{-8+21}{27} = \frac{13}{27}.$

(b) Compute $\mathbb{E}[X^2] = (-1)^2\frac{3+1+1+3}{27} + 3^2\frac{3+4}{27} = \frac{8+63}{27} = \frac{71}{27}.$

(c) Compute $\mathbb{E}[X^2Y] = 1\frac{1}{27} + 2\frac{1}{27} + 3\frac{3}{27} + 27\frac{4}{27} = \frac{120}{27}.$

3. If $j < 0$, $f_Y(j) = \mathbb{P}\{Y = j\} = 0$. If $j > 0$, then

$$f_Y(j) = \mathbb{P}\{Y = j\} = \mathbb{P}\{Y - 20 = j\} = \mathbb{P}\{Y = 20 + j\} = p(1-p)^{j+19}.$$

If $j = 0$, then

$$\begin{aligned}
 f_Y(0) &= \mathbb{P}\{Y = 0\} = \mathbb{P}\{10 \leq Y \leq 20\} = \sum_{j=10}^{20} p(1-p)^{j-1} = p(1-p)^{10} \sum_{j=0}^{10} (1-p)^j \\
 &= p(1-p)^{10} \frac{1 - (1-p)^{11}}{1 - (1-p)} = (1-p)^{10} \{1 - (1-p)^{11}\}.
 \end{aligned}$$

If $1 \leq j \leq 9$, then

$$\begin{aligned} f_Y(j) &= \mathbb{P}\{Y = j\} = \mathbb{P}\{10 - X = j\} + \mathbb{P}\{X - 20 = j\} \\ &= \mathbb{P}\{X = 10 - j\} + \mathbb{P}\{X = 20 + j\} = p(1 - p)^{9-j} + p(1 - p)^{19-j}. \end{aligned}$$

Thus,

$$f_Y(j) = \begin{cases} 0 & \text{if } j < 0 \\ (1 - p)^{10}\{1 - (1 - p)^{11}\} & \text{if } j = 0 \\ p(1 - p)^{9-j} + p(1 - p)^{19-j} & \text{if } 1 \leq j \leq 9 \\ p(1 - p)^{j+19} & \text{if } j \geq 10 \end{cases}$$

Math 361, Section F1, Spring 2002
Exam 3, April 26

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 100 Points

1. 30 points Let X be a continuous uniform random variable on $(0, 2)$; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } 0 < t < 2 \\ 0 & \text{else.} \end{cases}$$

Define the function

$$\varphi(u) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{u} & \text{if } 0 < u \leq 1 \\ 3 & \text{if } 1 < u < 2. \end{cases}$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

- (a) 4 points Graph φ .
- (b) 9 points Compute $\mathbb{P}\{Y \leq 10\}$.
- (c) 9 points Compute $\mathbb{P}\{Y \leq 2\}$.
- (d) 8 points Compute F_Y , the cumulative distribution function of Y .
2. 65 points Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} 4st & \text{if } s \geq 1 \text{ and } 0 \leq t \leq \frac{1}{s^2} \\ 0 & \text{else.} \end{cases}$$

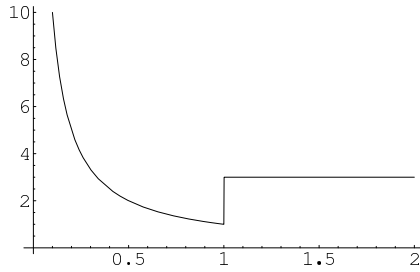
- (a) 25 points Compute f_X .
- (b) 10 points Compute $\mathbb{E}[X]$.
- (c) 30 points Compute $F_{X,Y}(s, t)$ for general s and t such that $s > 1$ and $t > 1$.
3. 5 points Suppose that X is a random variable such that

$$\mathbb{E}[e^{\theta X}] = e^{8\theta^2 - 5\theta}$$

for all $\theta \in \mathbb{R}$. What is the density of X ?

ANSWERS

1. (a) See Figure.



(b) $\mathbb{P}\{Y \leq 10\} = \mathbb{P}\{X \geq 1/10\} = 1 - \frac{1}{20}$.

(c) $\mathbb{P}\{Y \leq 2\} = \mathbb{P}\{\frac{1}{2} \leq X \leq 1\} = \frac{1}{20}$.

(d)

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2}(1 - \frac{1}{t}) & \text{if } 1 \leq t < 3 \\ 1 - \frac{1}{2t} & \text{if } t \geq 3 \end{cases}$$

2. (a)

$$f_X(s) = \int_{t=-\infty}^{\infty} f_{X,Y}(s, t) dt = \begin{cases} \int_{t=0}^{1/s^2} 4st dt & \text{if } s \geq 1 \\ 0 & \text{else} \end{cases} = \begin{cases} \frac{2}{s^3} & \text{if } s \geq 1 \\ 0 & \text{else} \end{cases}$$

(b)

$$\mathbb{E}[X] = \int_{s=-\infty}^{\infty} sf_X(s) ds = \int_{s=1}^{\infty} \frac{2}{s^2} ds = 2.$$

(c)

$$F_{X,Y}(s, t) = \int_{u=-\infty}^s \left\{ \int_{v=-\infty}^t f_{X,Y}(u, v) dv \right\} dv$$

For $s \geq 1$ and $t < 1/s^2$,

$$F_{X,Y}(s, t) = \int_{u=1}^s \left\{ \int_{v=0}^t 4uv du \right\} dv = \int_{u=1}^s \left\{ \int_{v=0}^t 4uv du \right\} dv = t^2(s^2 - 1).$$

For $s \geq 1$ and $t \geq 1/s^2$,

$$F_{X,Y}(s, t) = \int_{u=1}^s \left\{ \int_{v=0}^{1/u^2} 4uv du \right\} dv = 1 - \frac{1}{s^2}.$$

3. X is Gaussian with mean -5 and variance 16 ; i.e.,

$$f_X(t) = \frac{1}{\sqrt{32\pi}} \exp \left[-\frac{t^2}{32} \right]$$

for all $t \in \mathbb{R}$.

Math 361, Section F1, Spring 2002
Final, May 8

SHOW ALL WORK TO QUALIFY FOR FULL CREDIT.

Maximum possible score: 150 Points

1. 30 points There 7 balls labelled A through G in a box. We pick them out of the box one by one.
- (a) 10 points What is the probability that the fourth ball is C ?
- (b) 10 points What is the probability that the fourth ball is C and the fifth ball is D ?
- (c) 10 points What is the probability that the fourth ball is C given that the fifth ball is D ?

2. 30 points Suppose that X is a discrete random variable with density of the form

$$f_X(j) \stackrel{\text{def}}{=} \begin{cases} \frac{c}{2^{|j|}} & \text{if } j \text{ is an integer and } |j| \geq 1 \\ 0 & \text{else} \end{cases}$$

for some constant c which we will determine. Define two new random variables

$$S \stackrel{\text{def}}{=} \frac{X}{|X|} \quad \text{and} \quad M \stackrel{\text{def}}{=} |X|;$$

in other words, S , which takes values in $\{-1, 1\}$, is the sign of X and M is the magnitude of X .

- (a) 5 points Determine the constant c .
- (b) 10 points Compute the density of S
- (c) 10 points Compute the density of M
- (d) 5 points Are S and M independent?
3. 40 points Suppose that X and Y are independent discrete random variables with

$$f_X(j) = \begin{cases} \frac{1}{10} & \text{if } j \in \{1, 2, \dots, 10\} \\ 0 & \text{else} \end{cases}$$
$$f_Y(j) = \begin{cases} \frac{1}{20} & \text{if } j \in \{1, 2, \dots, 20\} \\ 0 & \text{else} \end{cases}$$

Define $Z_1 \stackrel{\text{def}}{=} X + Y$ and $Z_2 \stackrel{\text{def}}{=} \max\{X, Y\}$.

- (a) 20 points Compute f_{Z_1} , the density of Z_1 .
- (b) 20 points Compute $\mathbb{P}\{Z_2 \leq j\}$ for all integers j .

4. 20 points Let X be a continuous uniform random variable on $(-1, 1)$; i.e.,

$$f_X(t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} & \text{if } -1 < t < 1 \\ 0 & \text{else.} \end{cases}$$

Define the function

$$\varphi(u) \stackrel{\text{def}}{=} 1 - u^2$$

Define $Y \stackrel{\text{def}}{=} \varphi(X)$.

- (a) 2 points Graph φ .
- (b) 6 points Compute $\mathbb{P}\{Y \leq 1/2\}$.
- (c) 6 points Compute F_Y , the cumulative distribution function of Y .
- (d) 6 points If Y has a density compute it. If it does not have a density, tell me why.
5. 30 points Suppose that X and Y are continuous random variables with joint density

$$f_{X,Y}(s, t) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{s} e^{-t/s} & \text{if } t \geq s^2 \text{ and } s > 0 \\ 0 & \text{else} \end{cases}$$

- (a) 10 points Compute $\mathbb{P}\{Y \leq X\}$ (Hint: you may first want to identify the set $\{(s, t) : f_{X,Y}(s, t) > 0 \text{ and } t \leq s\}$).
- (b) 10 points Compute f_X , the density of X .
- (c) 10 points Compute $f_{Y|X}$, the conditional density of Y given X .

ANSWERS

1. (a) $\frac{{}^{(6)}_3 \cdot 1}{{}^{(7)}_4} = \frac{1}{7}$.
 (b) $\frac{{}^{(5)}_3 \cdot 1 \cdot 1}{{}^{(7)}_4} = \frac{1}{42}$.
 (c) $\left(\frac{1}{42}\right) / \left(\frac{{}^{(6)}_4 \cdot 1}{{}^{(7)}_5}\right) = \frac{1}{6}$.

2. (a)

$$1 = \sum_{k=-\infty}^{\infty} f_X(j) = 2c \sum_{j=1}^{\infty} \frac{1}{2^j} = c \sum_{j=0}^{\infty} \frac{1}{2^j} = 2c;$$

$$c = \frac{1}{2}.$$

- (b) We have

$$f_S(1) = \mathbb{P}\{X > 0\} = \sum_{j=1}^{\infty} \frac{1}{2^{|j|+1}} = \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2}$$

$$f_S(-1) = \mathbb{P}\{X < 0\} = \sum_{j=1}^{\infty} \frac{1}{2^{|j|+1}} = \frac{1}{4} \sum_{j=0}^{\infty} \frac{1}{2^j} = \frac{1}{2}.$$

- (c) For j a positive integer,

$$f_M(j) = \mathbb{P}\{S = j\} = \mathbb{P}\{X = j\} + \mathbb{P}\{X = -j\} = 2 \frac{1}{2^{|j|+1}} = \frac{1}{2^{|j|}}.$$

f_S is zero elsewhere.

- (d) Yes; for any positive integer j ,

$$f_{M,S}(j, 1) = \mathbb{P}\{M = j \text{ and } S = 1\} = \mathbb{P}\{X = j\} = \frac{1}{2^{|j|+1}} = f_M(j)f_S(1)$$

$$f_{M,S}(j, -1) = \mathbb{P}\{M = j \text{ and } S = -1\} = \mathbb{P}\{X = -j\} = \frac{1}{2^{|j|+1}} = f_M(j)f_S(-1);$$

both $f_{M,S}(j, s)$ and $f_M(j)f_S(s)$ are zero unless j is a positive integer and $s \in \{1, -1\}$.

3. (a)

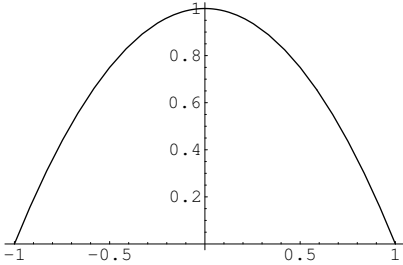
$$f_{Z_1}(k) = \sum_{k=-\infty}^{\infty} f_X(j-k)f_Y(k) = \frac{1}{200} \sum_{k=-\infty}^{\infty} \chi_{[1,10]}(j-k)\chi_{[1,20]}(k)$$

$$= \frac{1}{200} \sum_{k=-\infty}^{\infty} \chi_{[1,20] \cap [j-10, j-1]}(k) = \begin{cases} \frac{j-1-1+1}{200} & \text{if } j-1 \geq 1 \text{ and } j-10 \leq 1 \\ \frac{(j-1)-(j-10)+1}{200} & \text{if } j-1 > 10 \text{ and } j-1 < 20 \\ \frac{20-(j-10)+1}{200} & \text{if } j-10 \leq 20 \text{ and } j-1 > 20 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} \frac{j-1}{200} & \text{if } 1 \leq j \leq 10 \\ \frac{1}{20} & \text{if } 10 < j < 21 \\ \frac{31-j}{20} & \text{if } 21 \leq j \leq 30 \\ 0 & \text{else} \end{cases}$$

(b) For all j , $\mathbb{P}\{Z_2 \leq j\} = \mathbb{P}\{X \leq j\}\mathbb{P}\{Y \leq j\}$. For $j \leq 0$, $\mathbb{P}\{Z_2 \leq j\} = 0$. For $j \geq 20$, $\mathbb{P}\{Z_2 \leq j\} = 1$. For integers $j \in [1, 10]$, $\mathbb{P}\{Z_2 \leq j\} = \frac{j^2}{200}$, and for integers $j \in (11, 20)$, $\mathbb{P}\{Z_2 \leq j\} = \frac{j}{20}$.

4. (a) See figure.



(b)

$$\mathbb{P}\{Y \leq 1/2\} = \mathbb{P}\left\{1 - X^2 \leq \frac{1}{2}\right\} = \mathbb{P}\left\{X^2 \geq \frac{1}{2}\right\} = \mathbb{P}\left\{|X| \leq \frac{1}{\sqrt{2}}\right\} = 1 - \frac{1}{\sqrt{2}}.$$

(c)

$$\begin{aligned} F_Y(t) &= \mathbb{P}\{1 - X^2 \leq t\} = \mathbb{P}\{X^2 \geq 1 - t\} = \begin{cases} 0 & \text{if } 1 - t \geq 1 \\ \mathbb{P}\{|X| \geq \sqrt{1 - t}\} & \text{if } 0 < 1 - t < 1 \\ 1 & \text{if } 1 - t \leq 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } t \leq 0 \\ 1 - \sqrt{1 - t} & \text{if } 0 < t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} \\ &= \begin{cases} 0 & \text{if } t < 0 \\ 1 - \sqrt{1 - t} & \text{if } 0 \leq t < 1 \\ 1 & \text{if } t \geq 1 \end{cases} \end{aligned}$$

(d)

$$f_Y(t) = \begin{cases} \frac{1}{2\sqrt{1-t}} & \text{if } 0 < t < 1 \\ 0 & \text{else} \end{cases}$$

5. (a)

$$\begin{aligned} \mathbb{P}\{Y \leq X\} &= \mathbb{P}\{X \geq Y\} = \iint_{s>t} f_{X,Y}(s,t) ds dt = \int_{s=0}^1 \int_{t=s^2}^s \frac{1}{s} e^{-t/s} dt ds \\ &= \int_{s=0}^1 \{e^{-s} - e^{-1}\} ds = 1 - e^{-1} - e^{-1} = 1 - 2e^{-1}. \end{aligned}$$

(b) For $s < 0$, $f_X(s) = 0$. For $s > 0$,

$$f_X(s) = \int_{t=s^2}^{\infty} \frac{1}{s} e^{-t/s} ds = e^{-s}.$$

(c)

$$f_{Y|X}(t|s) = \frac{f_{X,Y}(s,t)}{f_X(s)} = \begin{cases} \frac{1}{s} e^{-t/s+s} & \text{if } t \geq s^2 \text{ and } s > 0 \\ 0 & \text{else} \end{cases}$$